Mathematics cop. 1
Direct

LAGRANGE'S COMPOUND PENDULUM

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1. *The Equations of Motion*

The so called polynomials of Laguerre were introduced into mathematical analysis by J. L. Lagrange[1] in his solution of a dynamical problem in which the oscillations of a vertical chain are represented approximately by those of a set of similar weights equally spaced on a light string which is vertical in the equilibrium position Oy. Let the weights be numbered 0, 1, 2, $\cdots$ in order, beginning with the lowest, and let x_n be the small horizontal displacement from the vertical line Oy of the weight W_n associated with the number n. If W is the magnitude of each weight, the tension in the portion $W_{n-1}W_n$ of the string is approximately equal to nW, for the cosine of the angle between the vertical and any portion of the string may be treated as unity when the oscillations are sufficiently small. Let a be the length of string between consecutive weights; then the angle between $W_{n-1}W_n$ and the vertical is approximately

$$(x_{n-1} - x_n)/a;$$

and so the equation of motion of the weight W_n is

$$\frac{W}{g}\frac{d^2x_n}{dt^2} = \frac{nW}{a}(x_{n-1} - x_n) - \frac{(n+1)W}{a}(x_n - x_{n+1}).$$

With a suitable choice of units the equation may be written in the form

$$x_n'' = n(x_{n-1} - x_n) - (n+1)(x_n - x_{n+1}), \tag{1}$$

where primes denote differentiations with respect to a modified time which will be denoted by τ. The actual relation between t and τ is

$$gt^2 = a\tau^2. \tag{2}$$

2. *Lagrange's solution of the equations of motion when the weight W_m is fixed.*

Let us suppose that the initial conditions are

$$x_n = a_n, \quad x_n' = 0, \quad \text{for } t = 0, \quad n = 0, 1, 2 \cdots. \tag{3}$$

Using the abbreviation

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}, \tag{4}$$

we introduce a set of quantities c_n, defined by the equations

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[1] *Miscellanea Taurinensia*, t. III. 1762–1765, *Oeuvres*, I. p. 534. The relation between the polynomials of Lagrange and Laguerre was, I think, first pointed out by Professor E. T. Whittaker after the present author had called his attention to a relation between the polynomials of Abel and Laguerre and his W-function. See Whittaker and Watson, *Modern Analysis*, p. 353.

$$c_0 = a_0,$$
$$c_n = n!\left[a_0 - \binom{n}{1}a_1 + \binom{n}{2}a_2 - \cdots + (-1)^n a_n\right], \tag{5}$$

and an operator ω with the property that

$$\omega^n c_q = c_{n+q}, \quad n = 0, 1, 2, \cdots. \tag{6}$$

A solution of the equations (1) and the boundary conditions (3) is then represented symbolically by

$$x_n = L_n(\omega)\cos(\tau\sqrt{\omega})c_0, \tag{7}$$

where

$$L_n(z) = 1 - \binom{n}{1}\frac{z}{1!} + \binom{n}{2}\frac{z^2}{2!} - \cdots (-)^n\frac{z^n}{n!} \tag{8}$$

is the polynomial of Lagrange and Laguerre.[2]

The verification is immediate because this polynomial satisfies the difference equation

$$n[L_{n-1}(z) - L_n(z)] - (n+1)[L_n(z) - L_{n+1}(z)] = -zL_n(z) \tag{9}$$

and the set of equations (5) gives

$$a_n = L_n(\omega)c_0. \tag{10}$$

In the particular case when $c_n = u^n c_0$, where u is a constant, we have

$$x_n = L_n(u)\cos(\tau\sqrt{u})c_0 \tag{11}$$

and $x_m = 0$ *for all values of* t if

$$L_m(u) = 0. \tag{12}$$

This is the result given by Lagrange. It is known that the m roots of the equation (12) are all positive and unequal.[3] We shall denote them by $u_1, u_2, \cdots, u_m$ and shall suppose that these numbers are arranged in order of magnitude, u_1 being the smallest and u_m the largest.

The solution for the initial conditions

$$x_n = a_n, \quad x_n' = b_n \quad \text{for} \quad \tau = 0, \quad n = 0, 1, 2 \cdots m-1, \tag{13}$$

and the requirement that $x_m = 0$ for all values of t, is

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[2] The notation used here is that adopted by E. Hille in the Proceedings of the National Academy of Sciences, vol. 12 (1926), pp. 261, 265, 348, and by G. Szegö in Mathematische Zeitschrift, vol. 25 (1926), p. 87.

[3] This was proved by Laguerre, Bulletin de la Société Mathématique de France, vol. 7 (1879). *Oeuvres de Laguerre*, vol. 1, p. 428. The distribution of the roots is discussed by E. R. Neumann in Jahresbericht der Deutschen Mathematiker Vereinigung, Band 30 (1921), p. 15.

$$x_n = \sum_{s=1}^{m} [A_s \cos(v_s\tau) + B_s \sin(v_s\tau)] L_n(u_s), \tag{14}$$

where $u_s = v_s^2$ and

$$\sum_{s=1}^{m} A_s L_n(u_s) = a_n, \tag{15}$$

$$\sum_{s=1}^{m} v_s B_s L_n(u_s) = b_n. \tag{16}$$

These equations can be solved for A_s and $v_s B_s$ if the determinant

$$D \equiv |L_p(u_q)| \qquad (p = 0, 1, \cdots, m-1, q = 1, 2 \cdots m)$$

is different from zero. If D were zero it would be possible with suitable coefficients $C_0, C_1, \cdots C_{m-1}$ to construct a polynomial

$$F_{m-1}(x) = \sum_{r=0}^{m-1} C_r L_r(x)$$

with the m roots, $x = u_1$, $x = u_2$, $\cdots$, $x = u_m$. This however, is impossible because the roots are all distinct and the polynomial is only of degree $m-1$.

If the string is supported from a fixed point dividing the line $W_{m-1}W_m$ in the ratio $\alpha:\beta$, there is a solution of a similar type; but the equation for the determination of the quantities u_q is now of type $G(x) = 0$, where

$$G(x) = \alpha L_{m-1}(x) + \beta L_m(x) = 0, \tag{17}$$

α and β being positive constants. The polynomial of Laguerre satisfies the equation

$$xL_m'(x) = mL_m(x) - mL_{m-1}(x), \tag{18}$$

where the prime now denotes a differentiation with respect to x. This equation tells us that if x has a positive value for which $L_m(x) = 0$ the functions $L_m'(x)$ and $L_{m-1}(x)$ have opposite signs, m being positive. Now $L_m'(x)$ has opposite sign at two consecutive roots of $L_m(x)$ and so $L_{m-1}(x)$ has also opposite signs at two consecutive roots of $L_m(x)$. This shows that a root of the equation $G(x) = 0$ lies between two consecutive roots of the equation $L_m(x) = 0$; indeed there cannot be three or more roots in such an interval because this would give the equation $G(x) = 0$ too many roots, since we must allow at least one for each interval. Allowing just one root for each interval there is till one root left. To locate this we observe that if μ is the greatest root of the equation $L_m(x) = 0$, $L_m'(\mu)$ and $L_{m-1}(\mu)$ have opposite signs and so $L_m(x)$ and $L_{m-1}(x)$ have opposite signs for $x > \mu$. Eventually $|L_m(x)|$ is enormously large compared with $|L_{m-1}(x)|$ and so $G(x)$ must vanish for some value of x greater than μ. The roots of the equations $L_m(x) = 0$, $G(x) = 0$ can thus be placed in correspondence in such a way that

each root of the second equation is greater than the corresponding root of the first. The physical meaning of this result is that the natural frequencies of the pendulum are increased by shortening the string. This is simply an illustration of a well known general theorem.

3. *Unlimited string*

When the string is unlimited in length a solution corresponding to the initial conditions (13) may be written down by making use of the fact that the function $L_n(u)$ satisfies the orthogonal relation (discovered by Abel and Murphy)

$$\int_0^\infty e^{-u}L_n(u)L_p(u)du = 0, \quad n \neq p, \tag{19}$$

$$= 1, \quad n = p.$$

The appropriate generalisation of (14) is thus

$$x_n = \int_0^\infty [A(u)\cos vt + B(u)\sin vt]L_n(u)du, \tag{20}$$

where $v = \sqrt{u}$ and $A(u)$, $B(u)$ are defined by the infinite series,[4]

$$A(u) = \sum_{p=0}^{\infty} a_p e^{-u}L_p(u). \tag{21}$$

$$vB(u) = \sum_{p=0}^{\infty} b_p e^{-u}L_p(u). \tag{22}$$

In particular, if $a_p = 0$, $p \neq s$, $a_s = 1$, $b_p = 0$, the solution is simply

$$x_n = \int_0^\infty e^{-u}\cos v\tau L_n(u)L_s(u)du. \tag{23}$$

The symmetry with respect to n and s should be noted. This is an illustration of Rayleigh's general reciprocal theorem.[5] Similarly, if $b_p = 0$, $p \neq s$, $b_s = 1$, $a_p = 0$, the solution is

$$x_n = \int_0^\infty e^{-u}\sin v\tau L_n(u)L_s(u)\frac{du}{v} \tag{24}$$

$$v = \sqrt{u}$$

and again there is symmetry.

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[4] The convergence of series of this type is discussed by Hille, l.c. See also G. Szegö, Mathematische Zeitschrift, vol. 25 (1926), p. 87.

[5] Proceedings of the London Mathematical Society, (1), vol. 4 (1873), p. 357; Scientific Papers, vol. 1, p. 179.

A solution equivalent to (23) may be obtained directly from (7) and may be written in the form

$$
\begin{aligned}
x_n = c_0 &- \binom{n}{1}\frac{c_1}{1!} + \binom{n}{2}\frac{c_2}{2!} - \cdots \\
&- \frac{\tau^2}{2!}\left[c_1 - \binom{n}{1}\frac{c_2}{1!} + \binom{n}{2}\frac{c_3}{2!} - \cdots\right] \\
&+ \frac{\tau^4}{4!}\left[c_2 - \binom{n}{1}\frac{c_3}{1!} + \binom{n}{2}\frac{c_4}{2!} - \cdots\right] \\
&- \cdots,
\end{aligned} \tag{25}
$$

the special values of the quantities a_p being introduced into the expression (5) for c_q. A similar expression for the integral in (24) may be obtained by integrating (25) from 0 to τ with respect to τ and replacing a_p' by b_p' in the expression for c_q.

4. *Relation to the Wave-Equation*

Let V be a solution of the equation

$$
\frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} = \frac{\partial^2 V}{\partial \tau^2} \tag{26}
$$

which arises from the two-dimensional wave equation when there is symmetry about an axis. Denoting this solution by the symbol $V(\rho,\tau)$ and assuming that it is finite and continuous for all positive values of ρ, we may form a definite integral

$$
x_n = \frac{1}{2^{2n+1}n!}\int_0^\infty e^{-\rho^2/4}\rho^{2n+1}V(\rho,\tau)\,d\rho \tag{27}
$$

which is readily seen to be a solution of the equation

$$
x_n'' = n[x_{n-1} - x_n] - (n+1)[x_n - x_{n+1}]. \tag{28}
$$

In particular, the wave-function

$$
V(\rho,\tau) = J_0(\rho\sqrt{u})\cos(\tau\sqrt{u}) \tag{29}
$$

gives rise to the 'Lagrangian' displacement

$$
x_n = e^{-u}L_n(u)\cos(\tau\sqrt{u}). \tag{30}
$$

When the string is supported at the weight W_m the condition $x_m = 0$ (for all values of τ) corresponds to the condition

$$
\int_0^\infty e^{-\rho^2/4}\rho^{2m+1}V(\rho,\tau)\,d\rho = 0 \tag{31}
$$

imposed on a solution of the wave-equation. This is a particular case of the more general condition

$$\int_0^\infty f(\rho^2)\rho d\rho V(\rho, \tau) = 0, \tag{32}$$

where $f(\rho^2)$ is a specified function. The special condition (31) is particularly interesting because a general type of wave-function V satisfying the requirements may be constructed with the aid of the polynomial of Laguerre and V is found to consist of the sum of m terms each of which is a function of ρ multiplied by a trigonometric function of τ.

The problem may be generalised in another way by considering the equation

$$\frac{\partial^2 V}{\partial \rho^2} + \frac{1 - 2m}{\rho}\frac{\partial V}{\partial \rho} = \frac{\partial^2 V}{\partial \tau^2}. \tag{33}$$

The differential equation satisfied by x_n is now

$$x_n'' = (n + m)(x_{n-1} - x_n) - (n + 1)(x_n - x_{n+1}). \tag{34}$$

This may be regarded as the equation of motion of the weight W_n when the weight of the string is taken into consideration. We shall indeed regard W as the weight of W_n and the portion of the string between W_{n-1} and W_n, while the weight of this portion of string alone will be supposed to be mW. A portion of string of weight mW will be supposed also to hang from the weight W_0. The tension of the string is then nW at W_{n-1} and $(n+m)W$ at W_n. The equation of motion is consequently of type (34). A solution of this equation may be expressed in terms of the generalised polynomial of Laguerre which was introduced into mathematical analysis by Sonine.[6] Using the notation of Hille we have[7]

$$\frac{1}{(1 - z)^{m+1}} e^{-xz/(1-z)} = \sum z^n L_n^m(x). \tag{35}$$

$$L_n^m(x) = \frac{1}{n!} e^x x^{-m} \frac{d^n}{dx^n}[x^{n+m} e^{-x}], \tag{36}$$

$$\int_0^\infty e^{-u} u^m L_n^m(u) L_s^m(u) du = 0 \quad (n \neq s)$$
$$= \frac{\Gamma(n + m + 1)}{\Gamma(n + 1)} \quad (n = s), \tag{37}$$

[6] Mathematische Annalen, vol. 16 (1880), p. 1.

[7] Loc. cit. The second equation is due essentialy to E. Kummer. See Crelle's Journal, vol. 15 (1836), p. 139, Eq. (4); the third equation is given by Sonine, loc. cit.; the fourth equation is ascribed by Szegö to É. Le Roy. Toulouse Annales, (2) vol. 2 (1900), p. 379. But Le Roy considers only the case $m=0$. The general formula is given on p. 103 of the author's *Electrical and Optical Wave Motion*, equation (222) and is also mentioned by Hille (loc. cit.).

$$e^{-u}u^{m/2}L_n^m(u) = \frac{1}{2^{2n+m+1}n!}\int_0^\infty e^{-\rho^2/4}\rho^{2n+m+1}J_m(\rho\sqrt{u})d\rho. \tag{38}$$

The solution of (33), which is represented by

$$V = \rho^m J_m(\rho\sqrt{u})\cos(\tau\sqrt{u}), \tag{39}$$

thus corresponds to the displacement

$$x_n = e^{-u}u^{m/2}L_n^m(u)\cos(\tau\sqrt{u}). \tag{40}$$

For the case of the unlimited string this solution may be generalised with the aid of the orthogonal relation (37) and the method of §3.

When the string is held at the point W_s the equation $x_s=0$ may be satisfied by choosing u so that

$$L_s^m(u) = 0. \tag{41}$$

Gegenbauer[8] has shown that this equation has s unequal positive roots.A geometrical proof for the special case $m=\frac{1}{2}$ is given by Bôcher.[9]

The foregoing interpretation of equation (34) seems to imply that $m<1$. This restriction can however, be avoided by supposing that the weights are buoyant instead of heavy. The case $m<0$ can be obtained by supposing the string to be buoyant and the weights heavy.

We may form a rough idea of the amplitude of the oscillation (40) when u is large by using Szegö's inequality (loc. cit.).

$$e^{-u/2}\left|L_n^m(u)\right| \leqq \binom{n+m+1}{n}, \quad u \geqq 0, \quad m \geqq -1, \tag{42}$$

or his alternative inequalities,

$$e^{-u/2}\left|L_n^m(u)\right| \leqq \binom{n+m}{n} \quad \text{for} \quad m \geqq 0, \quad u \geqq 0, \tag{43}$$

$$2 - \binom{n+m}{m} \quad \text{for} \quad -1 \leqq m \leqq 0, \quad u \geqq 0.$$

5. *Connection with the equation of heat conduction.*

If U satisfies the equation

$$\frac{\partial^2 U}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial\rho} = \frac{\partial U}{\partial\tau} \tag{44}$$

and is finite and continuous for all positive values of ρ, the definite integral

[8] Wiener Berichte (1887), p. 274; Amsterdam Proceedings, vol. 5 (1896–7), p. 185.

[9] Proceedings of the American Academy of Arts and Sciences, vol. 40 (1904), p. 469.

$$y_n = \frac{1}{2^{2n+1}n!}\int_0^\infty e^{-\rho^2/4}\rho^{2n+1}U(\rho,\tau)d\rho \tag{45}$$

is generally a solution of the equation

$$y_n' = (n+1)(y_{n+1} - y_n) - n(y_n - y_{n-1}), \tag{46}$$

which is satisfied by $y_n = e^{-u\tau}L_n(u)$, the corresponding function U being given by (38). Furthermore, the fundamental function

$$U = 1/\tau \exp\left[-\rho^2/4\tau\right] \tag{47}$$

gives rise to the simple solution

$$y_n = \tau^n/(1+\tau)^{n+1}. \tag{48}$$

This result may be generalised by taking U to be a solution of the partial differential equation,

$$\frac{\partial^2 U}{\partial \rho^2} + \frac{2m+1}{\rho}\frac{\partial U}{\partial \rho} = \frac{\partial U}{\partial \tau}. \tag{49}$$

The quantity y_n is then found to be a solution of the equation

$$y_n' = (n+1)[y_{n+1} - y_n] - (n-m)[y_n - y_{n-1}], \tag{50}$$

and the fundamental solution

$$U = \frac{1}{\tau^{m+1}}e^{-\rho^2/4\tau} \tag{51}$$

Gives rise to the simple solution

$$y_n = \tau^{n-m}/(1+\tau)^{n+1}. \tag{52}$$

6. *The torsional vibrations of a loaded shaft.*

The differential equation (1) occurs also in the theory of the torsional vibrations of a loaded shaft when the torsional stiffness of the portion of shaft between consecutive loads varies along the shaft in a suitable manner. Indeed, if θ_n is the angular displacement of the load W_n the equation of motion of this load is of type[10]

$$p_n\theta''_n = c_{n,n+1}(\theta_{n+1} - \theta_n) - c_{n-1,n}(\theta_n - \theta_{n-1}), \tag{53}$$

where p_n is an inertia coefficient and $c_{n,\,n+1}$ is the torsional stiffness of the portion W_nW_{n+1} of the shaft. We have only to take $c_{n,\,n+1}$ proportional to $(n+1)$ to obtain an equation reducible to (1).

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[10] See, for instance, J. Morris. "*The strength of shafts in vibration*" (Crosby Lockwood, London, 1929), p. 118.

A METHOD OF SOLVING A DETERMINATE SYSTEM OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS.[1]

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1. It is proposed to consider the system of simultaneous equations

$$(r) \qquad F_{r1}x_1 + F_{r2}x_2 + F_{r3}x_3 + \cdots + F_{rn}x_n = 0 \qquad (r = 1, 2, 3, \cdots n),$$

where $x_1, x_2, \cdots x_n$ are functions of an independent variable t, $F_{rs}(r, s = 1, 2, 3, \cdots n)$ are polynomials in D with constant coefficients, and D means d/dt.

There are at least three methods by which the full solutions of such a system can be obtained:—

(i) the ordinary method given in text-books on differential equations;[2]

(ii) a method (due to Chrystal)[3] for the reduction of the system to an equivalent diagonal system;

(iii) a method which does not seem to be given in text-books on differential equations, but which is dealt with at some length by Routh.[4]

It is proposed to discuss the third of these, and especially a case of failure which does not appear explicitly in Routh's work.

2. In any of these methods references must be made to the characteristic determinant of the coefficients in the system of equations:

$$\Delta(D) = \begin{vmatrix} F_{11} & F_{12} & F_{13} \cdots F_{1n} \\ F_{21} & F_{22} & F_{23} \cdots F_{2n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ F_{n1} & F_{n2} & F_{n3} \cdots F_{nn} \end{vmatrix}.$$

A complete theoretical treatment of such systems is given in Chrystal's paper referred to above, but, for most determinate systems of this type, the third method is the most direct and the most expeditious in practice, for it gives the full solution for each dependent variable with the proper number of arbitrary constants, without introducing any superfluous constants. It is, however, subject to certain cases of failure noted below, and, of these, the one which is the most difficult has been discussed by Routh.

The third method depends upon the use of cofactors (or first minors) of the elements in the determinant Δ. Let the given equations be numbered in order $(1), (2), \cdots, (n)$. Suppose that we use the $(n-1)$ equations $(1), (2), \cdots (r-1), (r+1), \cdots, (n)$ {i.e. all except equation (r)}, to determine the ratios

$$x_1 : x_2 : x_3 \cdots : x_n$$

[1] With special reference to an exceptional case (or case of failure) not noted by Routh.

[2] See, for example, reference 1 or 2 at the end of this article.

[3] See reference 3 or 4.

[4] See reference 5 or 6.

by a purely algebraical process, treating F_{pq}, which is a polynomial in D, as if it were a constant. The result may be written:

$$(r') \qquad \frac{x_1}{G_{r1}} = \frac{x_2}{G_{r2}} = \frac{x_3}{G_{r3}} = \cdots = \frac{x_n}{G_{rn}} = V,$$

where G_{rs} is the cofactor of F_{rs} in the characteristic determinant Δ. Substitution of these values of $x_1, x_2, \cdots x_n$ in the given equation (r) leads to the equation $\Delta(V)=0$.

If the roots of $\Delta(D)=0$ are all different, say $a_1, a_2, \cdots, a_k$, then

$$V = A_1e^{a_1t} + A_2e^{a_2t} + \cdots A_ke^{a_kt}.$$

If the roots of $\Delta(D)=0$ are not all different, the solution for V requires the usual modifications for repeated roots, but, in any case, it contains the full number (k) of arbitrary constants, where k is the degree of Δ as a polynomial in D. In the cases which usually arise, substitution in equation (r') then gives directly the full solutions for $x_1, x_2, \cdots x_n$ which are equivalent to the given system of equations. It will be noted that no arbitrary constants other than $A_1, A_2, \cdots A_k$ can appear in these solutions and that, with the exception of certain cases considered below, the method indicated obtains these solutions as directly as possible.

3. *Exceptional cases (or cases of failure).* If $\Delta(D)$ contains a factor $D-a$, there is in the solution for V a term of the type A_0e^{at} and, of course, if $\Delta(D)$ contains the factor $(D-a)^\lambda$, where λ is a positive integer, there are terms of the type

$$[A_0 + A_1t + A_2t^2 + \cdots + A_{\lambda-1}t^{\lambda-1}]e^{at}.$$

Now it is possible that all the cofactors of the r^{th} row in the characteristic determinant $\Delta(D)$, i.e. $G_{r1}, G_{r2}, \cdots G_{rn}$, may contain the factor $D-a$. In this case the arbitrary constant A_0 will be missing from the solutions for each of the variables $x_1, x_2, \cdots, x_n$. This exception is noted by Routh,[5] who points out that "if all the minors of only one row vanished, we could find the values of $x_1, x_2, x_3, \cdots$, etc., by choosing as our operators the minors of some other row. But this cannot be done if all the minors of all the rows are zero." Routh then gives a full discussion of the case when all the first minors contain $(D-m)^\beta$, all the second minors $(D-m)^\gamma$, etc., while $\Delta(D)$ contains $(D-m)^\alpha$. He shows how the full solutions may be obtained, including what he calls solutions of double type, of triple type, etc.

It is obvious that, whatever the values of $\alpha, \beta, \gamma, \cdots (\alpha>\beta>\gamma>\cdots)$, the above exceptional case is one in which $\Delta(D)=0$ has repeated roots. There is, however, an exceptional case of a similar character (not mentioned explicitly by Routh and apparently not noticed by him), in which the roots of $\Delta(D)=0$

[5] See reference 5, p. 164 or reference 6, p. 214.

may be all different, though, as will be explained below, there are at least two simple methods of obtaining the full solutions and so avoiding the apparent difficulty of this case. Thus if all the cofactors of the first row contain a factor $(D-a_1)$ all the cofactors of the second row a factor $(D-a_2)$, and so on, where $a_1, a_2, a_3, \cdots$ are all different, the complete solutions for $x_1, x_2, \cdots, x_n$ cannot be obtained from any single equation of the type (r') and yet $\Delta(D)=0$ may not have any repeated factors. In this case Routh's remark that "we could find the values of x_1, x_2, x_3, etc., by choosing as our operators the minors of some other row" is scarcely justified as it stands, though it is easy to see (as in the example below), that it may be justified if it be extended to include a combination of the results obtained from two different rows, i.e. from two equations (r') and (s'), $(r \neq s)$. A similar result holds if the common factors of the different rows are of the types

$$(D-a_1)^{\lambda_1},\ (D-a_2)^{\lambda_2},$$

or even combinations of these such as

$$(D-a_1)^{\lambda_1}(D-a_2)^{\mu_1};\quad (D-a_2)^{\lambda_2}(D-a_3)^{\mu_2};\quad (D-a_1)^{\lambda_3}(D-a_3)^{\mu_3};$$

etc., provided always that no common factor such as $(D-a_1)$ occurs in all the minors. By taking a combination of the results from an appropriate set of equations of type (r') it must always be possible to obtain the full solutions for $x_1, x_2, \cdots, x_n$.

Example.

$$D(D+2)x + (D+2)y + (D+2)z = 0; \tag{1}$$

$$x + (D+2)y + z = 0; \tag{2}$$

$$(D+1)x + (D+1)y + D(D+1)z = 0. \tag{3}$$

Solving for the ratios $x:y:z$ from (2) and (3),

$$\frac{x}{(D+1)(D^2+2D-1)} = \frac{y}{-(D+1)(D-1)} = \frac{z}{-(D+1)^2} = V_1. \tag{1'}$$

Similarly, from (1) and (3):

$$\frac{x}{(D+1)(D+2)(D-1)} = \frac{y}{-(D+1)(D+2)(D^2-1)} = \frac{z}{(D+1)(D+2)(D-1)} = V_2. \tag{2'}$$

From (1) and (2),

$$\frac{x}{-(D+2)(D+1)} = \frac{y}{-(D+2)(D-1)} = \frac{z}{(D+2)(D^2+2D-1)} = V_3. \tag{3'}$$

On substituting from equation (r') in equation (r) for any one of the cases $r=1, 2, 3$, V_r is found to satisfy the equation $\Delta(V)=0$, where Δ is the characteristic determinant of the given system of equations.

Thus

$$\Delta = D(D+1)(D+2)(D+3)(D-1).$$

Hence

$$V = A + Be^{-t} + Ce^{-2t} + Ee^{-3t} + Fe^{t},$$

and contains the full number (five) of arbitrary constants which must remain in the final solutions for x, y, z in this system of equations. If will be noted that $\Delta(D)$ has no repeated factors, and also that the cofactors (denominators) in equations (1′), (2′), (3′) contain the common factors

$$(D+1), (D+1)(D+2)(D-1), (D+2),$$

respectively. Thus if we denote by x_r, y_r, z_r the actual solutions obtained from equation (r'), $r=1, 2, 3$,

the solutions x_1, y_1, z_1, will contain no term e^{-t};

" " x_2, y_2, z_2 " " " " e^{-t}, e^{-2t}, or e^{t};

" " x_3, y_3, z_3 " " " " e^{-2t}.

The actual solutions are readily found in the forms:

$$(\alpha)\quad \begin{cases} x_1 = -A + Ce^{-2t} - 4Ee^{-3t} + 4Fe^{t} \\ y_1 = A - 3Ce^{-2t} - 8Ee^{-3t} \\ z_1 = -A - Ce^{-2t} - 4Ee^{-3t} - 4Fe^{t} \end{cases}$$

$$(\beta)\quad \begin{cases} x_2 = -2A - 8Ee^{-3t} \\ y_2 = 2A - 16Ee^{-3t} \\ z_2 = -2A - 8Ee^{-3t} \end{cases}$$

$$(\gamma)\quad \begin{cases} x_3 = -2A - 2Ee^{-3t} - 6Fe^{t} \\ y_3 = 2A + 2Be^{-t} - 4Ee^{-3t} \\ z_3 = -2A - 2Be^{-t} - 2Ee^{-3t} + 6Fe^{t}. \end{cases}$$

No single set of solutions (α), (β), or (γ) can provide the full (i.e. the most general) solutions of the given system of equations, but these full solutions may be obtained by taking a combination of (α) and (γ). It will be noted that these full solutions cannot be obtained from the combination (α) and (β), since the denominators in (1′) and (2′) all contain the factor $(D+1)$, and this causes terms in e^{-t} to be absent from sets (α) and (β). For similar reasons a combination of (β) and (γ) does not give any term in e^{-2t}.

Taking the combination (α) and (γ) and putting the arbitrary constants in as simple forms as possible, the most general solutions of the system of equations may be given as:

$$x = A + Ce^{-2t} + Ee^{-3t} + Fe^{t}.$$

$$y = -A + Be^{-t} - 3Ce^{-2t} + 2Ee^{-3t}.$$

$$z = A - Be^{-t} - Ce^{-2t} + Ee^{-3t} - Fe^{t}.$$

4. *An alternative method of dealing with the exceptional case* noted above may be illustrated by means of the preceding example. This method may be described as a combination of the first and third methods of paragraph 1. More explicitly, it uses the first (or ordinary) method with its excessive number of arbitrary constants to supply just that part of the solution which must be absent when, in the third method, the cofactors of any one row have a common factor. But it is important to notice that the first method as a whole is not required, and it is used only for that portion of the solution (usually small) where it is known that the third method must be deficient.

Thus in the preceding example x_1, y_1, z_1 may be found as before by using equation (1′) and, since all the denominators in (1′) contain the factor $(D+1)$, these solutions cannot contain any term in e^{-t}.
Hence put

$$x = x_1 + He^{-t}; \quad y = y_1 + Ke^{-t}; \quad z = z_1 + Le^{-t}.$$

Now x_1, y_1, z_1, satisfy the given equations, so that He^{-t}, Ke^{-t}, Le^{-t} taken separately must also satisfy these equations.
Substituting in (1) and (2),

$$-H + K + L = 0; \qquad H + K + L = 0.$$

[(3) gives only $0=0$].
Hence

$$H = 0; \qquad L = -K.$$

The solutions now obtained can easily be put in the same forms as those given above.

Perhaps it may be added that this alternative method is successful also in dealing with the exceptional case noted by Routh and so avoids his solutions of double type, triple type, etc. Thus in the example[6] constructed by Routh to illustrate his methods, each cofactor on the first row contains a factor $(D-1)^3$. Hence after finding x_1, y_1, z_1 as before we can put

$$x = x_1 + (A_1 + B_1t + C_1t^2)e^t,$$

etc., and proceed to the construction of the full solutions of the system.

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ON THE NUMERICAL SOLUTION OF A BOUNDARY VALUE PROBLEM

By W. E. MILNE, University of Oregon

The approximate solution of linear boundary value problems has been attacked by various methods, such as the method of Ritz,[1] depending in principle on the calculus of variations, the method of least squares,[2] the method of least mth powers,[3] and the method of constant coefficients.[4] The object of this note is to show how the method of numerical integration may be applied to the determination of the characteristic values of λ belonging to the homogeneous differential system

$$(d^2u/dx^2) + G(x, \lambda)u = 0, \tag{1}$$

$$fu(a) + gu'(a) = 0, \quad f^2 + g^2 = 1, \tag{2}$$

$$hu(b) + ku'(b) = 0, \quad h^2 + k^2 = 1. \tag{3}$$

The function $G(x, \lambda)$ is assumed to be real and continuous and to possess a positive partial derivative with respect to λ for all real values of λ and for x in the interval $a \leqq x \leqq b$. It is further assumed that

$$\lim_{\lambda=-\infty} G(x, \lambda) = -\infty, \quad \lim_{\lambda=\infty} G(x, \lambda) = +\infty.$$

Then there exists[5] an infinite sequence of values of λ, $\lambda_0, \lambda_1, \lambda_2, \cdots$ for which (1), (2), and (3) have non-vanishing solutions. In the case in which $G(x, \lambda)$ is of the form $\lambda - q(x)$ the approximate value of λ_n can readily be determined by means of asymptotic formulas when n is large, but the first values, λ_0, λ_1, etc., (which are usually of the greatest physical importance) cannot be so easily determined. On the other hand the method here given is well adapted to the calculation of the smaller characteristic numbers. The particular case in which the interval is infinite has been treated in an earlier paper,[6] where an illustrative problem is solved by this method.

1. Let $u_1(x)$ and $u_2(x)$ be two solutions of (1) satisfying the conditions,

$$\begin{aligned} u_1(a) &= 1, \qquad u_2(a) = 0, \\ u_1'(a) &= 0, \qquad u_2'(a) = c > 0. \end{aligned} \tag{4}$$

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[1] Ritz, Crelle's Journal, vol. 135 (1909), pp. 1–61.

[2] See, e.g., Kryloff, Bulletin of the American Mathematical Society, vol. 32 (1926), pp. 346–350. Kryloff and Bogoliouboff, Bulletin of the Academy of Sciences, U.S.S.R. 1929, p. 471. Also Picone, Rendiconti del Circolo Matematico di Palermo, vol. 52 (1928), pp. 225–253.

[3] W. H. McEwen's *Thesis* (Unpublished).

[4] See for example Bogoliouboff and Kryloff, Bollettino dell' Unione Matematica Italiana, April, 1928.

[5] Cf. Birkhoff, Transactions of the American Mathematical Society, vol. 10, (1909), p. 264.

[6] Milne, Physical Review, vol. 35 (1930), pp. 863–867.

The constant c is arbitrary and may be chosen so as to make the calculation as simple as possible. We define $w(x)$ by the equation

$$w(x) = [u_1^2(x) + u_2^2(x)]^{1/2}$$

and easily show[7] that $w(x)$ satisfies the equation

$$(d^2w/dx^2) + G(x, \lambda)w = c^2w^{-3}, \tag{5}$$

with the initial values $w(a)=1$, $w'(a)=0$. The general solution of (1) can now be expressed by the formula

$$u(x) = Cw(x) \sin [\phi(x) - \theta_1], \tag{6}$$

in which

$$\phi(x) = c\int_a^x w^{-2}dx$$

and C and θ_1 are arbitrary constants. There is no loss in generality in setting $C=1$. To determine θ_1, we substitute $u(x)$ from (6) into (2) and obtain

$$\theta_1 = \tan^{-1}(cg/f).$$

In order to satisfy condition (3) we let

$$y(x) = hu(x) + ku'(x). \tag{7}$$

After substitution from (6) into (7), and after some transformations, we obtain

$$y(x) = U(x) \sin [\psi(x) - \theta_2], \tag{8}$$

in which

$$U^2(x) = [hw(x) + kw'(x)]^2 + k^2c^2w^{-2}(x),$$

$$\psi(x) = c\int_a^x [h^2 + k^2G(x, \eta)]U^{-2}(x)dx, \tag{9}$$

and

$$\theta_2 = \tan^{-1}(ck/h) - \tan^{-1}(cg/f).$$

It is clear from (7) and (8) that the necessary and sufficient condition that λ be a characteristic number is that

$$\psi(b) = \theta_2 + n\pi \quad (n \text{ an integer}). \tag{10}$$

The quantity $\psi(b)$ is a function of λ only, and we now establish the important fact that if f, g, h, k and c are independent of λ, the function $\psi(b)$ is a continuously increasing function for all values of λ. To prove this we first note that $\psi(b)$ is independent of f and g. Now choose any value of λ, $\lambda=l$. Corresponding

[7] Milne, loc. cit.

to $\lambda = l$ there exists a solution $u(x)$ of (1) which satisfies (3). Then f and g may be determined so that this solution $u(x)$ of (1) also satisfies (2). Next let $x_0 < b$ be a value of x which varies with λ so that

$$hu(x_0) + ku'(x_0) = 0 \tag{11}$$

and such that, as x_0 approaches b, λ approaches l. Then by differentiation and substitution we obtain from (11),

$$dx_0/d\lambda = -[u'(\partial u/\partial\lambda) - u(\partial u'/\partial\lambda)][u^2 + u'^2]^{-1}[h^2 + k^2G(x_0, \lambda)]^{-1}. \tag{12}$$

We have also the equation [obtained from (1) by differentiating with respect to λ, and integrating with respect to x, after eliminating $G(x, \lambda)$ $(u\partial u/\partial x)$]

$$u'(\partial u/\partial\lambda) - u(\partial u'/\partial\lambda) = \int_a^x (\partial G/\partial\lambda)u^2 dx, \tag{13}$$

and finally, since $\psi(x_0) = \theta_1 + n\pi$, we have

$$(\partial\psi/\partial\lambda) + (\partial\psi/\partial x_0)(dx_0/d\lambda) = 0. \tag{14}$$

From (12), (13) and (14), together with (9), it follows that

$$\partial\psi/\partial\lambda = cU^{-1}[u^2 + u'^2]^{-1}\int_a^x [\partial G/\partial\lambda]u^2 dx,$$

which is always positive. This holds in particular when $x_0 = b$, $\lambda = l$, and therefore establishes the fact that $\psi(b)$ is an increasing function of λ.

2. We now show how the foregoing results enable us to determine characteristic numbers by numerical integration. We select k values of λ, namely $\lambda^{(1)}$, $\lambda^{(2)}$ $\lambda^{(3)}$, $\cdots$, $\lambda^{(k)}$ (usually equally spaced for convenience) chosen so that the desired characteristic number lies somewhere between $\lambda^{(1)}$ and $\lambda^{(k)}$. For each of these values of λ we solve equation (5) by numerical integration. Then using in turn each of the solutions of (5) now obtained we substitute in the formula

$$N = (1/\pi)\left\{c\int_a^b [h^2 + k^2G(x, \lambda)][(hw + kw')^2 + k^2c^2w^{-2}]^{-1}dx - \theta_2\right\} \tag{15}$$

and determine N for each by numerical quadrature, obtaining k values $N_1 < N_2 < \cdots < N_k$. If a characteristic value of λ lies between $\lambda^{(1)}$ and $\lambda^{(k)}$, there will be an integer $N = n$ between N_1 and N_k. The corresponding value of λ can be determined by inverse interpolation and will be one of the desired characteristic numbers. It is clear that θ_2 can be determined so that the characteristic number corresponding to $N = n$ will be precisely λ_n.

3. For certain special types of boundary conditions the integral (13) assumes particularly simple forms; for example in the case of the conditions

$$fu(a) + gu'(a) = 0, \quad u(b) = 0,$$

we have

$$N = (1/\pi)\left\{ c \int_a^b w^{-2} dx - \theta_2 \right\},$$

and similarly of course for the conditions

$$u(a) = 0, \quad fu(b) + gu'(b) = 0.$$

Likewise for the conditions

$$fu(a) + gu'(a) = 0, \quad u'(b) = 0$$

we have

$$N = (1/\pi)\left\{ c \int_a^b G(x, \lambda)[w'^2 + c^2 w^{-2}]^{-1} dx - \theta_2 \right\}.$$

4. It is evident from the definition of $w(x)$ that the value of N in (13) can be expressed in terms of the two solutions of (1), $u_1(x)$, $u_2(x)$, just as well as in terms of $w(x)$. Therefore one might obtain $u_1(x)$ and $u_2(x)$ by the process of numerical integration applied to equation (1), and then proceed as before. However, several of trial computations indicate that to obtain $w(x)$ from (5) rather than $u_1(x)$ and $u_2(x)$ from (1) results in a notable saving of time and labor.

A DISCUSSION OF A DIFFERENTIAL EQUATION

By CHARLES E. WILDER, Dartmouth College

Frequently the discussion of the families of line elements determined by a differential equation gives more information about the solution of a problem than does the formal solution of the differential equation. The following problem, proposed to Professor B. H. Brown by some doubtless well-meaning freshman, is an excellent example.

The Problem

An automobile moves along a straight road with a constant speed (v) while a man in a field beside the road walks with a constant speed (u) along such a path as to always keep a tree between him and the automobile. Determine his path.

The Discussion

First let us note that there is an obvious particular solution, namely, a straight line parallel to the road and on the opposite side of the tree at a distance from the tree equal to ua/v, where a is the distance from the tree to the road. The proof of this is a mere matter of similar triangles.

We shall set up the problem in both polar and rectangular coordinates. In Fig. 1.

O is the tree,

Q is the automobile at the instant (t),

P is the man at the same instant,

and the remainder of the notation is as shown in the figure. The time t is measured from the time of crossing the Y axis. From the figure we conclude that

$$\tan\theta = -vta^{-1} \text{ or } xy^{-1} = -vta^{-1},$$

and from this we obtain by differentiation:

(1) $$\sec^2\theta d\theta = -va^{-1}dt \quad \text{or} \quad (ydx - xdy)y^{-2} = -va^{-1}dt.$$

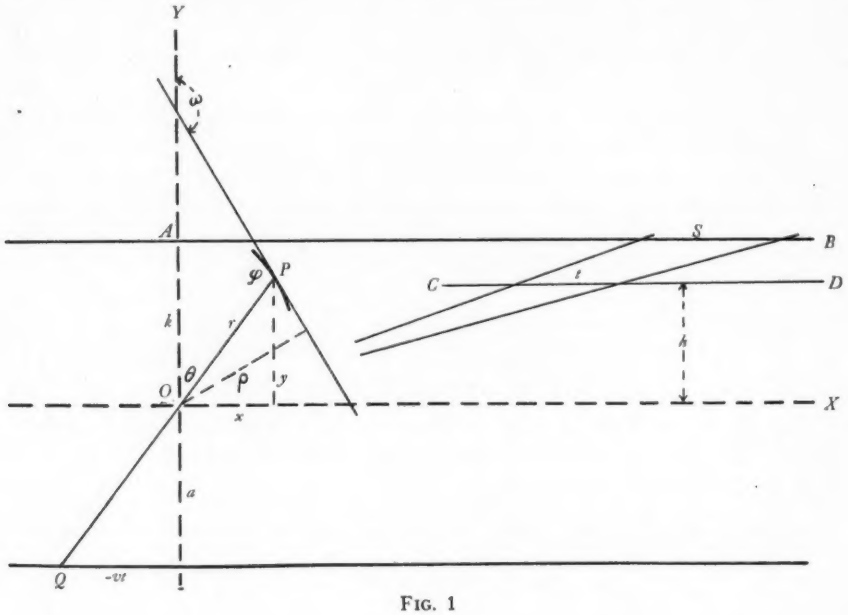

FIG. 1

If s is the length of arc along the path, $ds = udt$, and the substitution of dt from (1) gives

$$ds = -uav^{-1}\sec^2\theta d\theta \text{ or } ds = -uav^{-1}(ydx - xdy)y^{-2}.$$

Finally the appropriate formula for ds is used and the constant $-ua/v$ is replaced by k. This yields as the differential equation of the path

(2) $$(dr/d\theta)^2 + r^2 = k^2\sec^4\theta \text{ or } (y^4 - k^2x^2)(dy/dx)^2 + 2k^2xy(dy/dx) + y^2(y^2 - k^2) = 0.$$

The particular solution already noted may be written

$$r = k\sec\theta \quad \text{or} \quad y = k.$$

We should note that the differential equation does not require the tree to be between the man and the automobile.

The discussion will follow essentially the same method that is used in curve plotting. First we note that changing x into $-x$, or y into $-y$ does not change the equation. Hence if we can determine the solutions or parts of solutions in the first quadrant, the solutions in the remainder of the plane can be obtained by reflections in the axes. We do not want, however, any solutions below the axis of x, since these would not have the tree between the man and the automobile. Next we note that since the derivative is given in either system of coordinates by an expression containing a radical, there will be points of the plane through which there are no solutions and through such points as there are solutions there will be just two in general. If in the differential equation we set the radical equal to zero we obtain the equation of the curve, which we shall call the bounding curve, which bounds that part of the plane in which solutions lie. This curve is

$$r^2 = k^2 \sec^4\theta \text{ or } y^4 = k^2(x^2 + y^2).$$

It passes through the point $(0, k)$ and has points of inflection at

$$x = \pm \tfrac{1}{2}k\sqrt{3}, \quad y = \tfrac{1}{2}k\sqrt{6}.$$

If we use the ordinary methods of polar coordinates, the angle ϕ shown in Fig. 1 is given by

$$\cot\phi = r^{-1}dr/d\theta,$$

and from the differential equation we have

$$\cot^2\phi = (k^2 \sec^4\theta - r^2)r^{-2}, \tag{4}$$

which, being interpreted, means that the two solutions through any point make equal angles with the radius vector drawn to that point. In particular at points of the bounding curve both solutions are perpendicular to the radius vector while at the origin (the tree) there are two solutions for each direction of approach.

Since the two line elements at any point make equal angles with the radius vector their lines must both be tangent to the same circle about the origin. The radius ρ of this circle is the perpendicular distance from the origin to the line determined by the line element, and again referring to Fig. 1 we have

$$\rho = r \sin\phi;$$

and using the above expression for $\cot\phi$ we get

$$\rho^2 = r^4k^{-2}\cos^4\theta = y^4k^{-2}.$$

Thus we have the following construction for line elements. If P with coordinates (x, y) is any point, construct the circle with center at the origin and radius

$y^2/|k|$; then the line elements through P have the directions of the tangents to this circle from P. In particular the line elements along any horizontal are all tangent to the same circle.

We next inquire, where may solutions have horizontal or vertical tangents? This of course could be answered directly from the differential equation, but it is now easier to make use of the formula

$$\rho^2 = y^4 k^{-2}.$$

For if a line element is horizontal, $\rho^2 = y^2$; and we have

$$y^4 = k^2 y^2.$$

This gives us the axis of x which we can rule out at once, and the particular solution. We then conclude that no other solution except one through $(0, k)$ can have a horizontal tangent, for at no other point of the particular solution can the other solution as given by our construction be horizontal. If a line element is vertical, $\rho^2 = x^2$, and we have

$$y^4 = k^2 x^2,$$

which represents two parabolas through the origin whose axes are along the axis of x. At all points of these parabolas one of the two line elements is vertical.

After horizontal and vertical tangents come points of inflection. These we shall obtain from the polar coordinate condition which is

$$r^2 - r^2(d^2r/d\theta^2) + 2(dr/d\theta)^2 = 0.$$

When the derivative and second derivative are found from the differential equation and substituted in this, after some reduction we obtain

$$r^2 = k^2 \sec^2 \theta,$$

which gives us only the particular solution. This time we can conclude, after due consideration of our method of construction for such line elements as lie near the particular solution, that no other solution can have a point of inflection.

We might note now that solutions can cross each other at right angles only when $\cot \phi = \pm 1$, which by using (4) gives

$$2r^2 = k^2 \sec^4 \theta \quad \text{or} \quad 2y^4 = k^2(x^2 + y^2),$$

a curve similar to the bounding curve. At each point of this curve the two line elements are perpendicular to each other.

If we desire to go still farther we can find the curvature of a solution at any point. To do this it is best to turn to the definition of curvature. Curvature is

$$C = d\omega/ds,$$

where, from Fig. 1, $\omega = \phi + \theta$. Hence

$$C = (d\phi/ds) + (d\theta/ds) = [1 + (d\phi/d\theta)](d\theta/ds).$$

Now from (4)

$$r^2 \csc^2 \phi = k^2 \sec^4 \theta,$$

and from this we derive by differentiation

$$(d\phi/d\theta) = 1 - 2 \tan \theta \tan \phi,$$

which is merely another form of our original differential equation. We already have, in the deduction of the differential equation,

$$(ds/d\theta) = k \sec^2 \theta,$$

and so we can write the curvature

$$C = 2(1 - \tan \theta \tan \phi) \cos^2 \theta / k.$$

In particular when $\theta = 0$, $C = 2/k$, so all solutions cross the axis of y with the same curvature which is that of the parabolas, $y^4 = k^2 x^2$ at the origin.

From the construction of line elements it is clear that there are solutions which remain below the particular solution as x increases. As we go out along such a solution the slope is always decreasing, as is easily seen from the method of construction of line elements. Hence the solution is asymptotic to the particular solution or else to a horizontal line below it. We can disprove the latter possibility by an argument suggested by Professor B. H. Brown.

In Fig. 1, AB is the particular solution and CD any other horizontal below it. Draw two radii from O and let the segments cut off by these on AB and CD be respectively s and t. Let h be the height of the line CD above the axis of x. Then

$$ts^{-1} = hk^{-1} < 1.$$

Now assume a solution asymptotic to CD. Then as we go out along the solution in the direction of increasing x the length of the segment cut on the solution by two near-by radii approaches the length cut by them from CD. That is, the ratio of the length cut from the solution to the length cut from AB approaches a number less than unity. But from the statement of our problem the lengths of the segments cut from any two solutions by two radii must always be equal. Hence there is no solution asymptotic to CD.

Similarly we can show that there is a family of solutions asymptotic to the particular solution from above.

While we are discussing asymptotic properties let us prove that our parabolas $y^4 = k^2 x^2$ are asymptotic to the bounding curve. Let (x_0, y_0) be a point on the bounding curve and (x_0, y_1) be on the parabola. We have

$$y_0^4 - k^2 y_0^2 = k^2 x_0^2$$

and

$$\begin{aligned} y_1^4 &= k^2 x_0^2; \\ \hline y_0^4 - y_1^4 &= k^2 y_0^2; \end{aligned}$$

and from this

$$y_0 - y_1 = k^2 y_0^2 (y_0^2 + y_1^2)^{-1}(y_0 + y_1)^{-1},$$

which shows that $y_0 - y_1$ approaches zero as we go out along the curve.

We are now ready to collect our various bits of information and draw our conclusions. In Fig. 2 the auxiliary curves are in full lines and solutions in dotted lines. At points of the bounding curve (a) line elements are constructed by means of the fact that they are perpendicular to the radius vector drawn to the point. Line elements at points of the particular solution (e) are on the tangents from the points to the circle about the origin of radius k. Any other

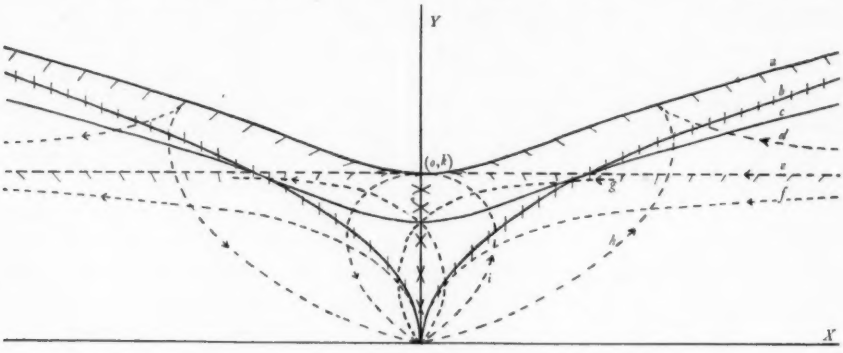

FIG. 2

a. boundary of region in which solutions lie.
b. parabola along which one family of line elements are vertical.
c. curve along which solutions cut at right angles.
d. sample solution asymptotic to particular solution from above.
e. particular solution.
f. discriminating solution.
g. sample solution asymptotic to particular solution from below.
h. solution of finite length.
i. discriminating solution.

If the automobile moves from left to right then the man follows solutions above in the directions shown by the arrow heads.

line elements desired may be drawn by means of our construction method. The fact that solutions do not have horizontal tangents or points of inflection helps us in joining up line elements into solutions.

Through the point $(0, k)$ passes the particular solution and another solution (i) which also has there the slope zero. This solution (i) may be approximately constructed by means of successive line elements or by numerical integration. If forms a closed curve as shown, entering the origin with slopes we shall call $\pm\lambda$. Through the origin are two solutions which are tangent to the axis of y. These also are constructed by line elements. In the figure, (f) is the one in the first quadrant.

Starting from the origin with any given slope are two solutions one of which curves up and the other down. If the given slope is less than λ that solution that curves down remains below (f) but is roughly similar to it and, like it, is asymptotic to the particular solution. On the other hand the solution that curves up has its slope constantly increasing till it crosses the parabola (b) vertically. It then proceeds to the bounding curve with a negative slope. An example of this type of solution is (h). If we trace the other solution (d) which strikes the bounding curve at the same point we find it is asymptotic to the particular solution from above. When we choose a slope at the origin greater than λ the solution that curves downward has the same character as before, but the other solution remains in the first quadrant within the solution (i), crosses the parabola vertically and then crosses the axis of y and finally goes off to the left asymptotic to the particular solution from below. These curves and their reflections in the y axis are the only solutions that cross this axis. The solution (g) and its reflection is the particular case in which the two line elements at the point in which they cross the axis of y are orthogonal. It is not difficult to see that all the line elements defined by the differential equation are used up by the various types of solutions we have considered so that our discussion is complete.

To connect the solutions of the differential equation thus found with our original problem we assume that the automobile moves along the road, which is of course, somewhere below the axis of x and parallel to it, from left to right. By means of lines through O (the tree) we can then determine the direction in which the man must move along the various solutions. These directions are shown in the figure by arrow heads. We now demand that the man make no abrupt changes in direction. If then the man starts initially from some point on the particular solution he can follow it indefinitely. If he starts in the first quadrant below (e) and outside of (i) he may go to the bounding curve where he must give up the game, or else to the tree where his only way out is to make an abrupt change in direction. If he starts between the particular solution and the bounding curve both solutions lead him to the bounding curve. If he starts inside of (i) he may go to the origin or he may continue indefinitely along one of the solutions on the left asymptotic to the particular solution from below. From any point in the second quadrant and below the bounding curve one of his two possible paths is a solution asymptotic to the particular solution and so may be followed indefinitely.

And now we turn back to the differential equation itself to see just what can be done by way of a proper solution. Contrary to all precedent known to us the first actual solution came by means of a contact transformation. Briefly, the parabola $y^2=kx$ seemed intrinsic in the problem and vague memories of some text suggested polar reciprocation in it. Thus we have the contact transformation

$$X = -(px - y)p^{-1}, \quad Y = \tfrac{1}{2}kp^{-1}, \quad P = \tfrac{1}{2}ky^{-1},$$

in which $p=dy/dx$. The use of this transformation on the differential equation yields

$$k^2 - 16P^4X^2 + 4Y^2 = 0$$

in which the variables are separable and the usual method leads to the solution. Incidentally the simpler transformation

$$X = p^{-1}, \quad Y = yp^{-1} - x, \quad P = y,$$

would have worked just as well.

Next we arrived at a solution from the differential equation in form (5):

$$d\phi/d\theta = 1 - 2 \tan\theta \tan\phi.$$

The substitution,

$$z = \tan\phi, \quad \tau = \tan\theta,$$

gives

$$(dz/d\tau) = (1 + z^2)(1 + \tau^2)^{-1}(1 - 2\tau z),$$

which is of the form

$$dz/d\tau = P_0 + P_1 z + P_2 z^2 + P_3 z^3,$$

in which the P's are functions of τ. This form is preserved if we make the further substitution $z=au+b$, in which a and b are functions of τ. We then determine a and b so that the new P_0 and P_1 shall both be identically zero. This gives us

$$a = (1 + \tau^2)\tau^{-4}, \quad b = \tau^{-1},$$

b being a particular solution which we fortunately know. Our new equation is

$$du/d\tau = - 5\tau^{-4}u^2 - 2(1 + \tau^2)\tau^{-7}u^3,$$

and the substitution $u=v^{-1}$ reduces this to what may be called a normal form:

$$v(dv/d\tau) = 5v\tau^{-4} + 2(1 + \tau^2)\tau^{-7}.$$

Equations of this type can sometimes be solved. In this case the further substitution $s=\tau^2 v+\tau^{-1}$ reduces it to

$$d\tau/ds = \tfrac{1}{2}s(s^2 + 1)^{-1}\tau - \tfrac{1}{2}(s^2 + 1)^{-1}$$

which is linear in τ.

Lastly, of course, we found what seems to be the easiest method. Solve the original differential equation for x, thus:

$$x = [ky \pm y^2(1 + p^2)^{1/2}]k^{-1}p^{-1},$$

in which $p=dy/dx$. The sign merely determines the sense of s (the measure-

ment of arc) and we shall take it negative. Differentiate with respect to x and simplify to

$$0 = 2p^2(1 + p^2) - k(dp/dx)(1 + p^2)^{1/2} + y(dp/dx).$$

Since $(dp/dx) = p(dp/dy)$, this becomes

$$2p(1 + p^2)(dy/dp) = k(1 + p^2)^{1/2} - y,$$

or

$$(dy/dp) + \tfrac{1}{2}p^{-1}(1 + p^2)^{-1}y - \tfrac{1}{2}kp^{-1}(1 + p^2)^{-1/2} = 0,$$

and again we have an ordinary linear differential equation.

If the reader will take the trouble to finish up any or all of the methods of solution suggested he will see why the formal solution is a useless form of answer to our given problem. We refrain from writing them out. They would take too much space.

THE PROBABILITY FUNCTION

By N. R. WILSON, University of Manitoba

The determination of the even probability function is a basic mathematical problem for students in statistics and in engineering and other exact sciences, but the mathematics involved places it quite beyond the average of these groups. Though receiving attention since the time of Gauss and Laplace, methods of derivation leave much to be desired in the matter of rigor, and have been severely criticized by an eminent economist.[1] Grouping together Gauss's first proof in his *Theoria Motus* and others in the elementary treatment of statistics making no mathematical pretensions, derivations fall into four classes. The commonest follows the lines of Gauss's second proof, based on the assumption that the "most probable value" is the average.[2] Other assumptions are that the function can be expanded in an infinite Maclaurin series and is always positive, with minor assumptions eight in number.[3] The first assumption has always been questioned. Also while it follows from the definition of probability that the function cannot be negative, experimental evidence certainly does not justify discarding zero values. This is the method especially attacked by Keynes (loc. cit., p. 208); the real criticism, however, being what is read into the misleading term "most probable value." The other two methods refer more particularly to the law of error in measuring lengths. Of these Hagen's[4] is based on the as-

[1] J. L. Keynes, *Probability*, pp. 205–214.

[2] Gauss, *Theoria Comb. Obs.*; Merriman, *Least Squares*, p. 22; Leland, *Least Squares*, p. 211; Coolidge, *Probability*, p. 113.

[3] Coolidge, l.c., p. 118.

[4] Hagen, *Grundzüge der Wahrscheinlichkeitsrechnung*, p. 31; Merriman, *Least Squares*, p. 17.

sumption that these errors are the resultant of infinitesimal errors obeying the binomial law. The other, due to Lord Kelvin, appears to have passed unnoticed.[5] This is based on the assumption that, if carried to two dimensions, points located by the measured lengths will lie symmetrically about the true position. The assumption is justified by experience with falling stones; evidence hardly bearing very directly on the measurement of lengths.

Probability is a mathematical concept—subjective, that is to say—and the derivation in question should, so far as possible, be made to depend upon other mathematical concepts rather than upon objective considerations. The method last cited above suggests a method of so doing which materially simplifies the discussion. The additive law for the probability of the alternative occurrence of independent constituent events, and the multiplicative law for the concurrence of such events, are usually regarded as theorems, not as assumptions. Whether such or not, they are taken as true here, and the solution, essentially contained in equation (2) below, rests on them.

A variable t is said to be determined by the even law of probability, $\phi(t^2)$, when, in the usual phrasing of applied mathematics, the probability that t lies between t and $t+dt$ is $\phi(t^2)dt$. The function ϕ is then subject to the condition following from the definition of probability, (*i*) of being not negative and (*ii*) of satisfying the relation

$$\int_{-\infty}^{\infty} \phi(t^2)dt = 1. \tag{1}$$

With reference to (*i*), the discussion is simplified by assuming as is usually done that ϕ is positive. We make this assumption in the body of the text, indicating in brackets how it may be avoided. We add the analytic assumptions that ϕ and its first derivative are continuous.

Consider a continuum of points $P(x, y)$ located in a plane with reference to a pair of rectangular axes, QX and QY; the variables x and y being determined by the laws of probability $\phi(x^2)$ and $\phi(y^2)$, respectively. Consider now the laws of probability for the determination of the coordinates (x', y') of the same set of points with reference to a set of axes, QX' and QY', bisecting the angles between the former pair. From the complete symmetry of the set with reference to the new axes, it follows that the law is even and the same for both axes, say $f(x'^2)$ and $f(y'^2)$.

[Analytically, integrating for the probability, fdx', that a point lies within a strip of width dx', parallel to QY', we have

$$f = \int_{-\infty}^{\infty} \phi(u^2)\phi(v^2)dy',$$

where $u=(x'+y')/\sqrt{2}$, $v=(x'-y')/\sqrt{2}$. By inspection, f is unchanged on

[5] Thomson & Tait, *Natural Philosophy*, Part I, p. 449.

reversing the sign of x, proving its evenness; and becomes the same function of y on interchanging the axes. The existence of this integral follows from equation (1) since neither $\phi(u^2)$ nor $\phi(v^2)$ is negative. It may be added that this is the only place where condition (i) does not appear unnecessary, being elsewhere deducible from equation (1).]

By the multiplicative law, the probability that a point lies in the usual infinitesimal neighborhood of $P(x, y)$ is $\phi(x^2)\,\phi(y^2)dx\,dy$, and in that of $P(x', y')$ is $f(x'^2)\,f(y'^2)dx'\,dy'$. Equating the probabilities per unit area that it lies in the neighborhood of $P(x, y; x', y')$, we have

$$\phi(x^2)\phi(y^2) = f(x'^2)f(y'^2).$$

Putting $y=0$ in equation (2), so that $x'=-y'=x/\sqrt{2}$ we have

$$a\phi(x^2) = \{f(x^2/2)\}^2, \tag{3}$$

where $a=\phi(0)$; whence $A=f(0)$.

[To show that $a=\phi(0)$ is positive, we cannot have $f(0)=0$. For if so, putting $x'=0$ in (2), so that $y=-x$, we have $\phi(x^2)=0$ for all values of x, contradicting equation (1). Since from (2) $\phi(0)=\pm f(0)$, we have $\phi(0)\neq 0$. From (i), ϕ cannot be negative. Hence $\phi(0)$ is positive. For a similar reason the relation $\phi(0)=\pm f(0)$, leads to $f(0)=a$.]

Reversing the above process—i.e. starting with the continuum $P(x', y')$, where x' and y' are determined by the laws of probability $f(x'^2)$ and $f(y'^2)$, and examining the laws along OX and OY—we obtain a relation of the form

$$af(x'^2) = \{g(x''^2/2)\}^2,$$

where $g(x''^2)$ gives the law along OX, x'' denoting temporarily the coordinate along this axis. Since $x''=(x'+y')/\sqrt{2}=x$, the arguments in the deduced law $g(x''^2)$ along OX and the original law $\phi(x^2)$ are the same. Hence g and ϕ are identical, or

$$af(x'^2) = \{\phi(x'^2/2)\}^2. \tag{4}$$

Eliminating f between (3) and (4),

$$a^3\phi(x^2) = \{\phi(x^2/4)\}^4.$$

Replacing x by $x/2^n$, where $n=1, 2, 3, \cdots$, and eliminating intermediate values of the arguments, we have, on writing m for 4^n,

$$a^{m-1}\phi(x^2) = \{\phi(x^2/m)\}^m. \tag{5}$$

Writing $x^2=u$, $\log \phi(u)=\psi(u)$, and differentiating,

$$\psi'(u) = \psi'(u/m).$$

[To legitimize these steps when zero values of ϕ are not excluded by hypothesis, suppose that $\phi(x^2)=0$ for some value of x. Then by (5) $\phi(x^2/m)=0$ for

all positive integral values of n, m being 4^n. Since ϕ is continuous, $\phi(0) = \lim \phi(x^2/m)$ as $n \to \infty$ and therefore $m \to \infty$; or $\phi(0) = 0$, contradicting the second paragraph (which is in brackets.]

Since $\phi'(u)$ is continuous and $\phi(u)$ positive, $\psi'(u)$ is continuous. Hence, on taking the limit of (6) as $n \to \infty$ and therefore $m \to \infty$, we have $\psi'(u) = \psi'(0)$, or

$$\phi'(u)/\phi(u) = \psi'(0). \tag{7}$$

Since $\phi(x^2)$ is positive for all values of x^2, the existence of the integral in equation (1) requires that ϕ decrease for some value of x^2, and therefore that $\phi'(u)$ should be negative or zero for the same value of u. Rejecting the zero value as it leads to a constant value of ϕ, we may write $-h^2$ for $\psi'(0)$ in (7). Integrating this, $\phi(x^2)$ is of the form $ke^{-h^2x^2}$. Substituting this value of ϕ in the relation

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x^2)\phi(y^2)dxdy = 1,$$

deducible on inspection from equation (1), and expressing the area integral in terms of polar elements of area, we have

$$2\pi k^2 \int_0^{\infty} e^{-h^2r^2} r dr = 1, \text{ where } r^2 = x^2 + y^2;$$

whence $k = h/\sqrt{\pi}$; or $\phi(x^2) = he^{-h^2x^2}/\sqrt{\pi}$, the form desired.

If we add the usual assumption that $\phi(x^2)$—and therefore $\log \phi(x^2)$—is a power series in x^2, the solution may be simplified, $\log \phi(x^2)$ being derived from the equation immediately preceding equation (5) by undetermined coefficients.

ON THE INVARIANCE OF THE DIVERGENCE OF A VECTOR FUNCTION

By REED LAWLOR, Los Angeles, California

Let $F(r) = F_1(x, y, z)i + F_2(x, y, z)j + F_3(x, y, z)k$ be a vector function of the vector $r = xi + yj + zk$, where i, j, and k are the unit vectors for a right-hand, orthogonal, Cartesian system of axes and where ∇F_1, ∇F_2, and ∇F_3 exist at each point of a domain D of three space. It is well-known that $\nabla \cdot F$, the divergence of F, has the form $(\partial F_1/\partial x) + (\partial F_2/\partial y) + (\partial F_3/\partial z)$ and that its value does not depend upon the choice of axes. The present note shows that $\nabla \cdot F$ has the above form when any three non-coplanar unit vectors i, j, and k are used. This seems of importance when one realizes that it is often desirable to use an oblique system of axes (for example, when a crystal is studied and the coordinate axes are chosen to coincide with those of the crystal) and it is very convenient to know that the divergence of a vector function has the same analytic form in

this system as in an orthogonal system. The curl of F does not have this property, neither does the gradient of a scalar function have the property.

Let a, b, and c be any three unit vectors such that $[abc]=a\cdot(b\times c)\neq 0$ and let i, j, and k be the unit vectors of a right-hand orthogonal system. Let $r=xi+yj+zk=ua+vb+wc$ be the vector from a common origin of both of the above systems to any point of D. Let

$$F(r)=F_1(x,y,z)i+F_2(x,y,z)j+F_3(x,y,z)k=f_1(u,v,w)a+f_2(u,v,w)b+f_3(u,v,w)c$$

be a vector function such that ∇F_1, ∇F_2, ∇F_3 exist at each point of D.

Theorem: $\nabla\cdot F=(\partial F_1/\partial x)+(\partial F_2/\partial y)+(\partial F_3/\partial z)=(\partial f_1/\partial u)+(\partial f_2/\partial v)+(\partial f_3/\partial w)$ at each point of D.

Proof:

$$\begin{aligned}(\partial F_1/\partial x) &= \nabla F_1\cdot i = i\cdot\nabla(F\cdot i) = i\cdot\nabla[f_1(a\cdot i)+f_2(b\cdot i)+f_3(c\cdot i)]\\ &= \nabla f_1\cdot(a\cdot i)i+\nabla f_2\cdot(b\cdot i)i+\nabla f_3\cdot(c\cdot i)i.\end{aligned}\tag{1}$$

Similarly[1]

$$(\partial F_2/\partial y) = \nabla f_1\cdot(a\cdot j)j+\nabla f_2\cdot(b\cdot j)j+\nabla f_3\cdot(c\cdot j)j,\tag{2}$$

$$(\partial F_3/\partial z) = \nabla f_1\cdot(a\cdot k)k+\nabla f_2\cdot(b\cdot k)k+\nabla f_3\cdot(c\cdot k)k.\tag{3}$$

Combining (1), (2), and (3), we obtain

$$\begin{aligned}\nabla\cdot F &= \nabla f_1\cdot[(a\cdot i)i+(a\cdot j)j+(a\cdot k)k]+\nabla f_2\cdot[(b\cdot i)i+(b\cdot j)j+(b\cdot k)k]\\ &\quad+\nabla f_3\cdot[(c\cdot i)i+(c\cdot j)j+(c\cdot k)k]\\ &= a\cdot\nabla f_1+b\cdot\nabla f_2+c\cdot\nabla f_3=(\partial f_1/\partial u)+(\partial f_2/\partial v)+(\partial f_3/\partial w),\end{aligned}$$

since $a\cdot\nabla f_1=(\partial f_1/\partial u)$, $b\cdot\nabla f_2=(\partial f_2/\partial v)$, $c\cdot\nabla f_3=(\partial f_3/\partial w)$. This completes the proof of the theorem and shows that $\nabla\cdot F=(\partial F_1/\partial x)+(\partial F_2/\partial y)+(\partial F_3/\partial z)$ for any choice of axes of x, y, and z so long as they are non-coplanar, where F_1, F_2, and F_3 are the respective components of F parallel to these axes.

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[1] A proof of the existence of ∇f_1 is as follows:

$$f_1=[Fbc]/[abc]=F_1[ibc]/[abc]+F_2[jbc]/[abc]+F_3[kbc]/[abc].$$

Hence ∇f_1 exists and

$$\nabla f_1=\nabla F_1[ibc]/[abc]+\nabla F_2[jbc]/[abc]+\nabla F_3[kbc]/[abc].$$

Similar proofs show the existence of ∇f_2 and ∇f_3.

NEW BOUNDS FOR THE ROOTS OF AN ALGEBRAIC EQUATION[1]

By E. C. WESTERFIELD, University of Colorado

§1.

From Descartes' rule of signs we know that the real equation

$$x^n = \sum_1^n p_r x^{n-r}, \quad p_r \geqq 0, \quad p_n > 0, \tag{1}$$

has exactly one positive root[2] and, since the left member of this equation is of higher degree than the right member, we have

LEMMA 1: *Any value of x satisfying the inequality*

$$x^n \leqq \sum_1^n p_r x^{n-r} \tag{2}$$

will form a lower bound for the positive root of equation (1).

LEMMA 2: *Any value of x satisfying the inequality*

$$x^n \geqq \sum_1^n p_r x^{n-r} \tag{3}$$

will form an upper bound for the positive root of (1).

Since every root of the equation

$$z^n + \sum_1^n b_r z^{n-r} = 0, \qquad b_r \text{ complex}, \tag{4}$$

must satisfy inequality (2) when (2) and (4) are related thru the equalities

$$x = |z|, \quad p_r = |b_r|, \qquad r = 1, \cdots, n; \tag{5}$$

it follows from Lemma 1 that the modulus of any root of (4) will form a lower bound for the positive root of equation (1). Stated differently we have[3]

LEMMA 3: *When equations* (5) *are satisfied the sole positive root of* (1) *will form an upper bound for the modulus of any root of* (4).

We now employ a simple identity to obtain a few special results. We write

$$(h+k)^n \equiv h(h+k)^{n-1} + k(h+k)^{n-1} \tag{6}$$

$$\equiv h(h+k)^{n-1} + hk(h+k)^{n-2} + k^2(h+k)^{n-2} \tag{7}$$

$$\equiv k(h+k)^{n-1} + hk(h+k)^{n-2} + h^2(h+k)^{n-2}. \tag{8}$$

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[1] Abstract from a thesis presented for the A. M. degree at the University of Colorado, June, 1930.

[2] G. Pólya und G. Szegö, Aufgaben und Lehrsätze aus der Analysis I, (1925), p. 87; Oscar Perron, Algebra II, Theorie der algebraischen Gleichungen, (1927), p. 20.

[3] See reference 2.

Continuing this process, we may choose either of the terms in (7) or (8) which contain the explicit expression $(h+k)^{n-2}$ and break it up into two terms of degree $(n-3)$ in $(h+k)$. Eventually the process will end with two terms of degree unity. Since at each step after the first, we must choose between two terms, it is evident that 2^{n-1} such expansions are possible. Half of these expansions may be obtained from the other half, however, by the interchange of h and k; and, since h and k enter symmetrically in the left member of (6), it follows that only 2^{n-2} of these expansions are distinct. These expansions will be of the general type

$$(9) \qquad (h+k)^n \equiv \sum_1^n P_r(h+k)^{n-r}, \qquad P_r \equiv P_r(h, k),$$

where the functions $P_r(h, k)$ are special polynomials determined in the manner indicated by (6), (7), and (8). Since such an expansion for $(h+k)^n$ is identical with the left member of (1) when $x=h+k$, it follows that when h and k are positive quantities so chosen that the coefficients of the right member of (9) are not less than the corresponding coefficients of the right member of (1),—then the value $x=h+k$ will satisfy (3) and, applying Lemma 2, will form an upper bound for the positive root of (1). Applying Lemma 3, this gives us

Theorem: *Every root of* (4) *must satisfy the inequality*

$$(10) \qquad |z| \leqq h+k,$$

where k and h are two arbitrary positive quantities so chosen that they satisfy each of the inequalities

$$(11) \qquad |b_r| \leqq P_r(h, k), \qquad r = 1, \cdots, n,$$

the n functions $P_r(h, k)$ being determined from one of the 2^{n-1} expansions of type (9).

One expansion of type (9) gives us

$$P_r = hk^{r-1}, \qquad r = 1, \cdots, n-1,$$
$$P_n = hk^{n-1} + k^n.$$

We see that $h=|b_1|$ and $k=\text{Max}\,|b_{r+1}:b_1|^{1/r}$, $r=1, \cdots, n-1$, satisfy (11); and applying the above theorem we find that every root of (4) must satisfy the inequality[4]

$$(a) \qquad |z| \leqq |b_1| + \text{Max}\,|b_{r+1}:b_1|^{1/r}, \qquad r = 1, \cdots, n-1.$$

Taking account of both terms in the expression for P_n, one obtains the more exact but much more cumbersome expression

$$(b) \qquad |z| \leqq |b_1| + \text{Max}\left\{|b_{r+1}:b_1|^{1/r}, [|b_n|:2]^{1/n}, |b_n:2b_1|^{1/(n-1)}\right\}, \qquad r = 1, \cdots, n-2.$$

[4] The fractional exponent, $1/r$, is used throughout to designate the positive rth root.

If, on the other hand, one takes $k=1$ and $h=\text{Max}\,(|b_r|,\ |b_n|-1)$, $r=1,\cdots,n-1$, (11) is again satisfied and one obtains

$$(c) \qquad |z| \leqq \text{Max}\,(1+|b_r|,\ |b_n|), \qquad r=1,\cdots,$$

These three simple expressions for the bounds are also obtained as corollaries to some theorems due to Kojima[5].

Another of these expansions gives us

$$\begin{aligned} P_1 &= h, \\ P_r &= k^2h^{r-2}, && r=2,\cdots,n-1, \\ P_n &= k^2h^{n-2}+kh^{n-1}. \end{aligned}$$

Applying the theorem as before, we may obtain the bound

$$(d) \qquad |z| \leqq |b_1| + \text{Max}\,|b_r : b_1^{r-2}|^{1/2}, \qquad r=2,\cdots,n,$$

while, as before, a more exact but much more cumbersome expression may be obtained by taking into consideration the second term in the expression for P_n.

Still another expansion of type (9) gives us

$$\begin{aligned} P_r &= hk^{r-1}, && r=1,\cdots,m-1, \\ P_m &= k^m, \\ P_r &= h^2k^{r-2}, && r=m+1,\cdots,n-1, \\ P_n &= h^2k^{n-2}+hk^{n-1}, \end{aligned}$$

where m may be any integer between 0 and n exclusive. When $m=n$ we have $P_m=P_n=k^n+hk^{n-1}>k^n$. For simplicity of notation we will now introduce the notation $q_r\equiv|b_r|$, q_r being positive. If, now, we take m to be the index of the greatest of the terms q_r and take k equal to this greatest term and h equal to the second greatest term, we see that (11) is satisfied. This gives us the simple bound

$$(e) \qquad |z| \leqq \text{Max}\,(q_r+q_s), \qquad r,s=1,\cdots,n,\ r\neq s.$$

Maintaining m as the index of the greatest of the terms q_r, we may again take $k=q_m$ and obtain the closer bound

$$(f) \qquad |z| \leqq q_m + \text{Max}\,(|b_r| : q_m^{r-1},\ [|b_s| : q_m^{s-2}]^{1/2}),$$
$$r=1,\cdots,m-1,\ s=m+1,\cdots,n.$$

These last two expressions may be compared with a simple bound

$$(g) \qquad |z| < 2\,\text{Max}\,(q_r), \qquad r=1,\cdots,n,$$

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[5] Tôhoku Mathematical Journal, vol. 5 (1914), p. 58.

obtained as corollary to a general theorem due to Fujiwara,[6] and also with a related bound obtained by Carmichael[7]

$$\text{(h)} \qquad |z| \leqq \sum_{1}^{n} q_r.$$

Table 1 gives some numerical upper bounds obtained by applying the eight formulae above to the three following polynomial equations:

$$\text{(A)} \qquad z^8 + 246z^7 - 305z^6 - 321z^5 + 355z^4 - 400z^3 + 420z^2 + 446z + 450 = 0,$$

$$\text{(B)} \qquad z^8 - 2z^7 + 38z^6 + 32z^5 - 160{,}000z^4 + 35{,}000z^3 - 4{,}200z^2 - 200z + 120 = 0,$$

$$\text{(C)} \qquad z^8 + z^7 + 10z^6 - 10z^5 + 700z^4 + 300z^3 + 100z^2 + 20{,}000z - 100{,}000{,}000 = 0.$$

TABLE 1

	(A)	(B)	(C)
(a)	247.3	45.1	14.9
(b)	247.3	45.1	13.6
(c)	447.0	160,001.0	100,000,000.0
(d)	263.5	202.0	25,001.0
(e)	263.5	28.1	15.2
(f)	263.5	22.1	11.0
(g)	492.0	40.0	20.0
(h)	285.3	47.4	30.9

§2.

Remembering that in equation (4) we have $b_0 = 1$, we may write a bound due to Carmichael and Mason[8] in the convenient form

$$\text{(i)} \qquad |z| < \left[\sum_{0}^{n} |b_r^2| \right]^{1/2}.$$

If we multiply (4) thru by $z-a$ (where a is an arbitrary complex quantity), we see that any bound for the roots of the resulting equation will also bound the roots of (4). Since, as with (4), the coefficient of the highest power term is unity, we have the resulting bound

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[6] Tôhoku Mathematical Journal, vol. 10 (1916), pp. 167–171.

[7] Bulletin of the American Mathematical Society, vol. 24 (1917–18), pp. 286–296.

[8] Bulletin of the American Mathematical Society, vol. 21 (1914), pp. 14–22; also, M. Kuniyeda, *Note on the roots of algebraic equations*, Tôhoku Mathematical journal, vol. 9 (1916), pp. 167–173; vol. 10 (1916), pp. 187–188; M. Fujiwara, *Ueber die Wurzeln der algebraischen Gleichungen*, Tôhoku Mathematical Journal, vol. 8 (1915), pp. 78–85; S. B. Kelleher, *Des limites des zéros d'un polynome*, Journal de Mathématiques, (7), vol. 2 (1916), pp. 169–171; and reference 2.

$$(12) \qquad |z| < \left[\sum_{0}^{n+1} |b_r - ab_{r-1}|^2 \right]^{1/2}, \qquad b_{n+1} = b_{-1} = 0.$$

Taking $a=1$ gives us a special result due to Williams[9]

$$(j) \qquad |z| < \left[\sum_{0}^{n+1} |b_r - b_{r-1}|^2 \right]^{1/2}, \qquad b_{n+1} = b_{-1} = 0.$$

We now seek to determine the arbitrary quantity a so as to make the bound determined through (12) a minimum. To this end we write

$$\begin{aligned} \sum_{0}^{n+1} |b_r - ab_{r-1}|^2 &= \sum_{0}^{n+1} (b_r - ab_{r-1})(\bar{b}_r - \bar{a}\bar{b}_{r-1}) \\ &= (1 + |a|^2) \sum_{0}^{n} |b_r^2| - 2 \sum_{1}^{n} R(b_r \bar{a} b_{r-1}) \\ &= (1 + |a|^2)A - 2R\left(\bar{a} \sum_{1}^{n} b_r \bar{b}_{r-1} \right), \qquad A \equiv \sum_{0}^{n} |b_r^2|, \\ &= (1 + \rho^2)A - 2\rho B \cos(\theta - \phi), \qquad \rho, B \geqq 0, \end{aligned}$$

where $\bar{a}$ denotes the conjugate of a, $R(\cdots)$ denotes the real part of the expression in the brackets, $\rho e^{i\phi} \equiv a$, and $Be^{i\theta} \equiv \sum_{1}^{n} b_r \bar{b}_{r-1}$. Considering this as a function of ρ and examining it for a minimum as ρ varies, we find that $\rho = (B:A)\cos(\theta - \phi)$ provides this minimum; and setting this value into the original expression we have

$$\sum_{0}^{n+1} |b_r - ab_{r-1}|^2 = A - (B^2:A)\cos^2(\theta - \phi).$$

Examining this expression as ϕ varies, we see that $\phi = \theta$ gives us a minimum. The resulting minimum bound is, therefore,

$$(k) \qquad |z| < \left[\sum_{0}^{n} |b_r^2| - \left| \sum_{1}^{n} b_r \bar{b}_{r-1} \right|^2 : \sum_{0}^{n} |b_r^2| \right]^{1/2},$$

and the value of a providing this minimum is given by

$$a = \sum_{1}^{n} b_r \bar{b}_{r-1} : \sum_{0}^{n} |b_r^2|.$$

Table 2 gives the numerical upper bounds obtained by applying (i), (j), and (k) to the following simple polynomial equations:

$$(D) \qquad z^5 + z^4 + z^3 + z^2 + z + 1 = 0,$$

$$(E) \qquad z^5 - z^4 + z^3 - z^2 + z - 1 = 0,$$

$$(F) \qquad z^5 - 4z^4 + 3z^3 - z + 4 = 0.$$

[9] Bulletin of the American Mathematical Society, vol. 28 (1922), pp. 394–6.

TABLE 2

	(D)	(E)	(F)
(i)	2.45−	2.45−	6.56−
(j)	1.41+	4.69+	11.22+
(k)	1.35+	1.35+	5.81−

For a comprehensive bibliography of literature on the location of roots of polynomials the reader is referred to an article by E. B. VanVleck in the Bulletin of the American Mathematical Society vol. 35 (1929), pp. 643–683.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y. and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

An Introduction to Mathematics, with Applications to Science and Agriculture. By I. L. Miller. New York, F. S. Crofts & Co., 1930. xiv+298 pages.

Contributions to the History of Determinants, 1900–1920. By Sir Thomas Muir. London, Blackie & Son, 1930. xxiv+408 pages. 30s net.

Theory of Functionals, and of Integral and Integro-Differential Equations. By Vito Volterra. London, Blackie & Son, 1930. xiv+226 pages. 25s net.

Advanced Algebra. By P. H. Graham and F. W. John. New York, Prentice-Hall, 1930. x+258 pages. $1.85.

Von Zahlen und Figuren. By Hans Rademacher and Otto Toeplitz. Berlin, Julius Springer, 1930. vi+164 pages.

Bericht über Neuere Untersuchungen und Probleme aus der Theorie der Algebraischen Zahlkörper. By H. Haase. Teil I: Klassenkorpertheorie; Beweise zu Teil I. 134 pages. 7.40 marks. Teil II: Reciprozitatsgesetz. iv+204 pages. 11.80 marks. Leipzig, B. G. Teubner, 1930.

Mathematics of Finance, preceded by Elementary Commercial Algebra. By B. E. Crenshaw, Z. M. Pirenian, and T. M. Simpson. New York, Prentice-Hall, 1930. xiv+384 pages, with tables and answers. $3.75.

Higher Arithmetic, designed for the use of high schools and colleges and for self-instruction. By Frederick A. Smith, C. E. Copyrighted, 1929, by the author. iii+138 pages, with portrait. $2.00.

Analytic Geometry. By A. M. Harding and G. W. Mullins. Revised Edition. New York, The Macmillan Company, 1930. x+316 pages. $2.50.

Introduction to Higher Geometry. By William C. Graustein. New York, The Macmillan Company, 1930. xvi+486 pages. $4.50.

Algebraic Charts. By Edgar Dehn. New York, The Nomographic Press, 1930. 14 sheets in a folder. $1.00.

Addition-Subtraction Logarithms, to Five Decimal Places. By L. M. Berkeley. New York, White Book and Supply Company, 1930. xii+136 pages. $3.25.

Plane and Spherical Trigonometry with Tables. By Leonard M. Passano. Revised Edition. New York, The Macmillan Company, 1930. $2.10.

Geometry Workbook. By H. B. Kingsbury and R. R. Wallace. Chicago, Bruce Publishing Company, 1930. 78 practice tests and 10 review tests. Paper, 76 cents.

The Logic of Discovery. By Robert D. Carmichael. Chicago, the Open Court Publishing Company, 1930. $2.00.

Debunking Science. By E. T. Bell. University of Washington Chapbooks, No. 44. Paper, 40 pages.

REVIEWS

Differential Geometry of Three Dimensions, Volume II. By C. E. Weatherburn Cambridge University Press, 1930. xii+240 pages. $4.25.

The second volume of this work on metric differential geometry continues the discussion of the subject along lines which are a natural extension of those followed in the first volume, which appeared in 1927. It seems appropriate, therefore, to make a few comments on certain characteristics which the two volumes have in common, before considering the second volume specifically.

The first thing that strikes the reader on turning the pages of the two volumes is that consistent use of vector analysis is made throughout. The classical notation of Gibbs is employed, symbols for vectors being printed in Clarendon type. Certain economies are thus effected in the way of simplifying and condensing the presentation of the subject. Those who have been in the habit of lecturing to graduate students on the subject of metric differential geometry, using the conventional methods, might do well to consider the advisability of trying out a presentation by vector methods. To anyone who ventures on this undertaking these two volumes before us will be very useful; they should be in the hands of the students as well as on the lecturer's desk.

Another commendable characteristic of the entire work is that the author has not allowed himself to forget that he is, before all, writing a treatise on geometry. The geometry is the thing that holds the center of the stage, and the analytical machinery is relegated to a subordinate place in the background where it ought to be in a book on geometry. The author does *not* make the mistake of becoming so immersed in the intricacies of his machinery, or so occupied with juggling his tools, that he loses sight of his main undertaking. Moreover, it is worthy of remark in this connection that the author's geometric insight and intuition are as clear and penetrating as his geometric interest is dominant.

The treatise lacks the profundity of such monumental works as those of Bianchi and Darboux. The more elementary parts of the subject are fairly adequately treated, but the more advanced portions are touched upon rather lightly. There is an abundance of good exercises. So the volumes are, we should

say, well suited for beginners, and would quite likely be very satisfactory as texts in the classroom, but would perhaps be only of secondary value on the reference shelf alongside Bianchi and Darboux.

There are certain features of the work which will perhaps not meet with unqualified approval. One of these is the entire omission of the Christoffel symbols. Another concerns a change in terminology. For reasons which seem compelling to the author, and which he explains in the preface to the first volume and more fully in a note at the end of the same volume, he insists on replacing the old terminology *mean curvature* and *total curvature* at a point of a surface by the expressions *first curvature* and *second curvature* respectively. We dissent from the author's opinion regarding this change in terminology, and believe, with a recent reviewer of the first volume in the Bulletin of the American Mathematical Society, that the total or Gaussian curvature should not "be subjected to the ignominy of the name 'second curvature.' "

Let us glance quickly at the contents of the first volume. We find treated here the most fundamental topics in the metric theory of curves, surfaces, and rectilinear congruences. The sequence in which the topics are arranged does not diverge very far from that which has become more or less conventional and is somehow reminiscent of Eisenhart's Differential Geometry, which has been the first avenue of approach to the subject for so many American students.

As we turn to the second volume and start through it, the reviewer gets an increasingly strong impression that we have here a collection of "selected topics" in metric differential geometry instead of a closely organized and coordinated whole. "Most of the chapters are based on recent papers by the author," as he himself asserts. There is, however, much other material included. There are thirteen chapters. After a preliminary chapter on differential invariants for a surface, two chapters are devoted to families of curves on a surface, and particularly, oblique trajectories. A fourth chapter is taken up with ruled surfaces and Weingarten surfaces. In Chapter V the author lays the foundations for a theory of curvilinear coordinates in ordinary metric space. He discusses in this connection the three-parametric gradient, divergence, and rotation as well as other differential invariants of functions in space which in recent times have entered largely into the theory of mathematical physics and here play also an important part in differential geometry. After a chapter on families of surfaces, and particularly Lamé families, the author discusses in Chapter VII dyadics, which are substantially tensors of the second order, and in the next chapter families of curves and functions of direction on a surface. Chapter IX is devoted to Levi-Civita's parallelism, especial attention being given to parallel displacement of a vector along a given curve on a surface; the results are applied to Tchebychef systems of curves on a surface. Next come three chapters on representation of surfaces, small deformations of curves and surfaces, and applicability of surfaces. The final chapter presents two methods of studying curvilinear congruences.

The book is well written. The author evidently understands the funda-

mental principles of good mathematical exposition. The typography is excellent. And it would not be surprising if some of the new portions of the material should prove to be stimulating to other writers and should thus lead to further new contributions to the literature of differential geometry.

ERNEST P. LANE

The Adjustment of Errors in Practical Science. By R. W. M. Gibbs. Oxford University Press, 1929. 112 pages.

This brief book is intended to serve as a practical guide to workers dealing with experimental and statistical data in other than the pure physical sciences. Among other topics it touches lightly on probability, least squares, give some detail on the normal curve of errors, and deals at some length with correlation theory. Numerous examples are worked out so that the methods of the text can be applied directly, although the treatment is non-mathematical, all rigorous proofs being reserved for the appendix of some twenty-seven pages Many graphs adequately illustrate the ideas which are expressed with clarity and simplicity. Beyond its expressed purpose this text may well interest the non-specialist in further study in the theory of errors and of statistics.

S. B. LITTAUER

Agricultural Mathematics. By L. C. Plant. McGraw-Hill Book Co. New York, 1930. x+200 pages. $2.50.

This book is written as a text for freshmen in agriculture and presupposes only the usual preparation in high school mathematics. It contains a large number of problems that are typical of those that the agriculturist will meet in his future study, each chapter has a group to which it is necessary to apply the principles developed in it, for example, balancing rations from a given list of feeds, or mixing a certain fertilizer from given materials furnish problems in the solution of simultaneous linear equations, while the capacity of a silo or the size of a drainage tile needed under given conditions furnish applications for quadratic equations. A whole chapter is given to the applications of probability to problems in heredity.

One of the longest chapters is "Statistics," in which enough of the fundamentals of the subject to enable the student to understand the statistical bulletins of the department of agriculture are supplied. Sufficient trigonometry to enable the student to solve triangles is contained in the last chapter.

The book is well written and has very few typographical errors for a first edition. However a few questions arose in the mind of the reviewer. Can the author justify the use of the word "transformations" as it occurs on page 9 where each of the steps in the solution of the equation, $ax+b=cx+d$, is spoken of as a transformation? On p. 45 the author defines $a^{-m}=1/a^m$, $a^{1/m}=\sqrt[m]{a}$, etc., without showing any reason for doing so. Is it not better to give the student the so-called proofs found in any college algebra or at least tell him why we

use these definitions? The explanation given of how to look up the logarithm of a number and more especially that for the converse problem seems to be inadequate for the average student.

In chapter III the author gives a brief but quite satisfactory discussion of the significant figures in a number obtained by measurement and of the computations with such approximate numbers. This feature might be added, with profit to the student, to other texts in freshman mathematics. The symbolic treatment of probability found on pp. 94–95 is quite good.

As a whole this text impresses the reviewer as being well adapted to use by the group of students for whom it was written. It should do much to popularize the study of mathematics with agricultural students, for the choice of problems in it furnishes the answer to the question, "Why should I study mathematics?", so often asked by this group.

HERMAN W. SMITH

A REVIEW OF A REVIEW

In the February, 1929 number of this Monthly there appeared a review by James Byrnie Shaw of Professor Keyser's book "The Pastures of Wonder" on which I believe comments should be made. It is my belief that this review fails to represent the book fairly, and also that it contains many statements that are at least open to question. For this reason I am asking the Monthly to give space for this rejoinder. In what follows excerpts from the review are in quotes.

"He has departed widely from his view, once expressed, that science is a sublimated form of play, the austere and lofty analogue of the kitten playing with the entangled skein or of the eagle sporting with the mountain winds." This figure of speech, containing as it does, and properly so, such general and vague words as "sublimated," "play," "austere," "lofty," and "analogue" should not, I believe, be regarded as the statement of a "view" as to how the word "science" should be delimited nor could Dr. Keyser have supposed that anyone might be fatuous enough to take it for such a statement. I, for one, do not find the slightest discrepancy between that figure of speech and the book reviewed.

"The book is a natural outcome of the notions of 'logistic' development by Russell and others. The antidote to this *disease* is . . .". (A "disease," is it?) I should say that the book—at least much of it—is a natural outcome of the work of all who have occupied themselves with the foundations of mathematics from Euclid down to the present. Is it fitting to characterize all this work as a *disease*?

"The premises may be false, they may be mere empty symbols [how gloriously confused this is], and the conclusion may be nowhere applicable, but if the logic is correct . . . we have a mathematical statement. It is the old and *absurd* statement of Russell, that in mathematics we do not know what we are talking about [we start with undefined symbols] nor if our conclusions are true [we start with unproved propositions]." The comments in brackets are mine. I confess that in my simplicity I have interpreted this "absurd" epigram so that

to me it seems quite sensible. Moreover, I know not a few who are regarded as leaders in present-day mathematics who interpret it and regard it pretty much as I do. The unqualified characterization of this statement of Russell's as "absurd" does not strengthen my confidence in the reviewer.

". . . the *empty* assertion that the conclusion follows from the premises." From what should the conclusion follow anyway? I look to Professor Shaw for enlightenment. How can a mathematical proposition be formulated to save it from being "empty"? But this goes too nearly to the heart of the matter to be treated with levity. How empty *are* such assertions? An example will help: Assuming the usual laws of motion and Newton's law of gravitation as our premises, the conclusion is that the planets move in certain orbits and at certain speeds. But suppose the planets do not move as thus predicted, as apparently has of late been shown to be the fact. Are the propositions involved no longer mathematical? I will venture that Dr. Shaw, in his long career as a teacher, has set problems on falling bodies under the (false) assumption that g is a constant and that he made some young person's happiness to depend upon the result thus called forth. The "premises were false" and "the results were nowhere applicable," but the resulting statements were certainly regarded as mathematical. How about the time honored *reductio ad absurdum*?

When the results from a set of premises prove by some other standard to be false, what do we do? Scrap our "logistic" structure? Nothing of the kind. We go back and refashion our premises. In this manner was *relativity* born. The purely deductive structure serves its purpose whether the conclusion squares with experience or not. What an amazing faith we have in the validity of this despised deductive logic! Rather than doubt its validity we change our most fundamental and intimate ideas about such matters as time, space, and motion! Does it not seem out of place to characterize propositions which lead to such final results as "empty" in a derogatory sense? It may not be far wrong to say that mathematical propositions serve us most profoundly when the "premises are false, and the conclusion nowhere applicable."

Does the conclusion follow from the premises? This "empty" question evidently occupied Euclid when he postponed, as long as he could logically, his consideration of parallels. It is very likely that he believed the sum of the angles of a triangle must in all cases be two right angles, but he could not make it follow from his premises until he added the axiom of parallels. Then he made the "empty" statement that the conclusion follows from *these* premises. If he had been less keen he might have made the categorical statement, presumably not "empty," that the sum is two right angles, "nilly willy." In that case Professor Shaw, along with hundreds of others, would make mention of Euclid's error when lecturing on the history of mathematics. Does the conclusion follow from the premises? This question is uppermost with those who are engaged in organizing mathematics into a consistent and adequate logical structure. Suppose we glance at the treatise on Projective Geometry by Veblen and Young. The beginning is made with an incomplete (non-categorical) set of axioms, and

a certain body of propositions is shown to follow from *these* axioms. Then another axiom is added to the "premises" and additional propositions are proved, and so on to the end of the chapter. Every proposition in this two-volume treatise is an "empty" assertion that a certain conclusion follows from certain premises. All is vanity saith the Lord!

"But he evidently fails to see that the forms [of propositions] themselves are objects. . . ." Certainly this is a subject of controversy and cannot be settled with a semi-impertinent "he evidently fails to see." In the last analysis it may be a question of taste whether the word "object" is to be used to include "form" in the sense in which it is used here. Moreover, if Mr. Shaw had read the book attentively, he would have noted that on page 59 the author had anticipated this point of the reviewer and rightly called it mere quibbling.

". . . one may well ask if the universe consists merely of propositions?" Certainly not, and equally certainly, Dr. Keyser does not claim that it does. As I read "The Pastures of Wonder" I find its purpose to classify propositions. The propositions of one class are of the form, p implies q (or, less accurately, in the "if-then" form). Such propositions are "proved" by showing the q is a logical consequence of p. These propositions are assigned to the field of mathematics. The conclusions (the q's) of such propositions are as valid as are the premises (the p's) and the logic employed in the proof, and the propositions are valid whether the premises are true of false. *Vide* the *reductio ad absurdum*.

Propositions whose validity rests on "objective tests," (this is my own brief interpretation of what I find in the book) Professor Keyser proposes to class as scientific as distinct from mathematical. There is nowhere in the book a suggestion that a treatise on mathematics contains (or should contain) nothing but hypothetical propositions, though our reviewer clearly implies that it does. I cannot quote for lack of space—see the last paragraph of the review. Professor Keyser is not urging the *absurd* thesis that the mathematician should occupy himself only with hypothetical propositions, though he may regard them as the framework, a skeleton if you will, which gives coherence and shape to the finished structure. Professor Keyser knows also that in the life history of a biological organism, the early stages are relatively crude in form—that the fundamental framework develops later, and reaches its final form only as maturity approaches. There may be an interesting analogy here but I refrain from following it further. One may be fully alive to the beauty of mathematics, and to a great many other things the reviewer mentions, and still agree entirely with Dr. Keyser's definition of mathematical propositions. In fact, I think Dr. Keyser himself is a conspicuous example.

"For all creative mathematicians are chiefly concerned in making categorical assertions about the ideal world in which they live." This "categorical" statement is, I believe, definitely false. They may be saying that this is "easy," or "difficult," or "lovely," or "detestable," and these assertions would all be categorical. But their mathematical propositions are hypothetical. Given a certain situation, with certain conventions, or axioms, so and so will follow.

The proof is invariably deductive. "Verification" by testing the result in special cases is not part of the proof.

It may be that ultimately propositions will be characterized by the type of proof on which they depend—according to their basis for validity—though this is closely connected with their form, the more completely the proposition is stated the more close is this connection. Whether or not Dr. Keyser's classification will be adopted just as it now stands, it seems fairly evident that some such classification is in the air. The role of deductive reasoning in thought is, I believe, due for a new type of recognition and attention. Whether we agree with Dr. Keyser on all points or not, he deserves our thanks and appreciation for producing "The Pastures of Wonder."

N. J. LENNES

Missoula, Montana, June 23, 1930.

MATHEMATICS CLUBS

All reports of club activities should be sent to Professor F. M. Weida, George Washington University, Washington, D. C.

CLUB TOPICS

MECHANICS—A DRAMATIC SKIT

By TOMLINSON FORT, Lehigh University

Dramatis Personae

1. Newtonian Mechanics, alias Brigham Young.
2. Physics
3. Chemistry
4. Astronomy
5. Engineering
6. Wave Mechanics
7. Matrix Theory
8. Relativity
9. Geometry

(The scene opens with an empty room. *Newtonian Mechanics* enters. He is a large man with rough dress, beard and deep masculine voice. Telephone rings. He goes to the phone.)

Newt. Mechs.: Yes, Yes. This is Young. Yes Brigham Young—B-R-I-G-H-A-M Y-O-U-N-G, Brigham Young. What is that? No, not always. That's right. That's what they used to call me. Newtonian Mechanics, from my father Isaac Newton. (Irritably) Why did I change my name? Well I got religion, that's why. If it were'nt for that I couldn't manage this family of mine. How are they? (Angrily) Go to Paradox! (Hangs up, moves over to another chair. Sits down.) Well I've had a day of it. By the Ghost of Galileo, I'm tired. (Relaxes). (Sits up and rings bell.)

(Enter the wives: (1) Physics who has recently had her face lifted. She is very frivolous. Her dress is covered with waves and lightning flashes. (2) Chemistry. She has a many-colored dress decorated with crucibles etc., light peroxide hair. She is sarcastic and cynical. (3) Astronomy. She is dignified and of lofty demeanor. Her dress is covered with stars. (4) Engineering. She is elderly. Her dress is that of an ordinary scrub woman decorated with a few faded levers or other machines. Her hair is slicked back and her sleeves rolled up. She is homely and practical. Each has her name on her breast.)

Physics: (With giggles and to no one in particular) You know I think I look much younger since my face-lifting operation. I. . . . .

Chemistry: (Interrupting) Really you do try to attract attention. Don't you?

(*Newtonian Mechanics* looks at them with resigned air. He takes out roll book and pencil.)

Newt. Mechs.: The wives will answer to roll call: Physics.

Physics: Present, b—u—t—.

Newt. Mechs.: (Roughly) Chemistry.

Chemistry: Here.

Newt. Mechs.: Astronomy.

Astronomy: (Resignedly) Why this eternal roll call. Have I not always been present, always inspired your best work? Do you not trust your old sweetheart? Come back to the enthusiasm of your youth. Do something for my perihelions.

Newt. Mechs.: Shut up! Engineering.

Engineering: You can count on me. If I were not here who would cook the dinner?

New. Mechs.: Well I'm ready for dinner now. Galileo knows I work hard enough for all of you.

Physics: (To herself). He has such beautiful hair.

Newt. Mechs.: You talk too much.

Physics: I must tell you of my electrons.

Newt. Mechs.: (Exasperated) For Newton's sake!

Chemistry: I don't feel so very well either."

Newt. Mechs.: Both of you take a dose of "law of motion oil" and go to bed. (Aside) That always fixes them.

(*Newt. Mechs.* gets up) I'll be back in a few minutes. No foolishness now. (Goes out).

Astronomy: (Sweetly) I can't help worrying about my perihelions. (*Physics* giggles. *Astronomy* assumes dignified look.)

Engineering: (Disgustedly) Smokestacks! (She begins to straighten things.)

Physics: (To *Chemistry*) But my dear you should see my new doctor. He has the most wonderful hair, all blonde and as if it had been marcelled. He thought that I was a debutante. He really did.

Chemistry: (Aside) Great Priestley, I wonder how I ever got into this family. Priestley knows old *Newtonian Mechanics* never did anything for me. (To *Physics*) Darling I should like to see this young man of yours.

(Door bell rings)

Physics: (Excitedly) I told him to call.

Engineering: Come in.

(Enter two up to the minute young men; both wear glasses and carry medical cases.)

Physics: How do you do? (To others) Allow me to present *Dr. Wave Mechanics.*

Chemistry and Astronomy: So glad to know you.

Engineering: (Slowly) Good evening.

Wave Mechs.: How do you do? Allow me to present *Dr. Matrix Theory.*

Matrix Theory: Pleased to meet you, I'm sure.

(*Physics* powders her nose; *Chemistry* uses bright red lipstick.)

Physics: My electrons, Doctor, don't you think an examination will be necessary?

Engineering: (Disgustedly) Smokestacks! (Goes to work at something.)

Wave Mechs.: (Paying no attention to *Engineering*) My beautiful young lady: I'm afraid so, and such a building-up process as you'll get. Come to our sanatorium.

Physics: (Aside to *Astronomy*) Notice his hair. Did you ever see such eyes?

Astronomy: My dear; remember our husband.

Physics: (To *Wave Mechanics*) Do you know our husband, *Newtonian Mechanics*?

Wave Mechanics: He is all wrong.

Matrix Theory: No, simply incompetent, but a trusting old soul.

Physics: He is old-fashioned but good-hearted. He married me when I was just a child. You know he is much older.

Matrix Theory: You should see our chief surgeon, *Mathematics*! There's a man for you! You should watch him operate. And our head nurse, *Mrs. Quantum*. It's really she that gives us our work to do.

Engineering: (Disgustedly) Smokestacks! I'll call *Newtonian Mechanics*. He will throw you upstarts out.

Physics: Please!

Chemistry: Possibly they can do *Physics* good. I'm all upset myself. Old *Newtonian Mechanics* never does anything for me. I sympathize with *Physics*.

Wave Mechs.: Physics, come with me. I'll look to your electrons myself, even your free electrons. You may have whatever you want. *Physics*, I love you.

Physics: I must pack my things.

Wave Mechs.: You need nothing. *Mrs. Quantum* will supply all.

Physics: I must have *Laws of Motion*.

Wave Mechs.: Leave them.

Physics: Let me take *Conservation of Matter*.

Wave Mechs.: No.

Physics: Boo hoo! Boo hoo! Please let me take *Conservation of Energy*. It is such a beautiful garment and can be worn on so many occasions. It is most economical.

Wave Mechs.: The most pernicious of them all. No! Come.

Physics: Please!

Astronomy: But she will be coming back soon. She is the wife of *Newtonian Mechanics*.

Wave Mechs.: No.

Physics: Boo hoo. Just a little *Conservation of Energy*.

Wave Mechs.: No.

Matrix Theory: Define energy and he will listen to you.

Physics: Boo hoo.

Chemistry: Define *Energy*! Who ever heard of such a thing?

Engineering: That's easy. My baby, *Work*, can do more than that. (All laugh.)

All: Isn't she old-fashioned? Ha Ha, Ha Ha.

(*Matrix Theory* and *Wave Mechanics* go out followed by *Physics*.)

(Enter *Newtonian Mechanics*.)

Newt. Mechs.: Where's *Physics*?

All: (in chorus) Gone.

Newt. Mechs.: Where?

Astronomy: With a handsome stranger.

Engineering: (with disgust) Smokestacks! Deserted her husband in the Lord for an upstart who called himself *Wave Mechanics* and another one who said he was *Matrix Theory*. Scandalous, I call it.

Astronomy: She loves him.

Chemistry: Yes and they promised to fix her electrons; Doctors, you know, and their head nurse is to look after all her little quanta.

Engineering: (with disgust) Smokestacks! If she ever has any.

Chemistry: She'll get description anyhow, and description is more than I get in this family.

Newt. Mechs.: The ungrateful woman! But I have the rest of you.

(Nervously) Answer to roll call.

(All answer, Here, but *Physics*.)

Newt. Mechs.: Let's have dinner. (The bell rings twice, *Astronomy* shivers, the door opens and a handsome stranger is there.

Newt. Mechs.: Who are you?

Relativity: A distant cousin of yours, *Relativity* by name, to help you with your work.

Newt. Mechs.: Welcome stranger. Wives, make room for *Relativity*. Here, Cousin. We must help *Engineering* with the dinner. Bring in the chairs.

Engineering: Smokestacks!

(*Newtonian Mechanics* and *Relativity* seize a chair together. *Relativity* drops it.)

Relativity: Oh my Universe. (Grabs his back) Sir, this is not the way. I stand here, you there, I on Mars, you on distant Neptune. I take a clock, you a vibrating atom.

Astronomy: Wonderful!

*Newt. Mechs.:*Nonsense!

Engineering: Smokestacks!

Relativity: We must have a plan for any chair, anywhere, without a care.

Newt. Mechs. and Engineering: Get to work or get out.

Relativity: Well cousin, you scorn my generality. No romance, no poetry, all triviality. You cannot even manage a perihelion. Adieu!

Engineering: Smokestacks! Get out. (Raises broom).

Astronomy: Hold, I must go with you.

All: What?

Chemistry: You his favorite wife, you his boasted sweetheart, Ha, Ha, Ha.

Astronomy: I must go. My perihelions.

Newt. Mechs.: (stutters, waves arms) S . . . . S . . . .

Relativity: Come. (Exit *Relativity* and *Astronomy*)

Chemistry: Well *Newt.* old boy, what do you think of that? Two gone. This family of saints is in a bad way. Ha, Ha, Ha.

Newt. Mechs.: By the ghost of Galileo! By my father *Newton!*

(*Engineering* begins to sweep violently)

Chemistry: Ha, Ha, Ha I'm tired of it. I'm going too. What have you ever done for me? Ha, Ha, Ha, you didn't know that your servant, Calculus, was my lover. Ha, Ha, Ha, Thought to hold us all, Ha, Ha, Ha. I have a new lover too, another servant of yours, *Differential Equations.* I'm going to *Differential Equations.* Ha, Ha. To *Differential Equations,* To *Differential Equations*! (Goes out)

(*Engineering* and *Newtonian Mechanics* stand looking at each other. *Mechanics* is unable to speak. He waves his arms and stutters.)

Engineering: It's all right, *Newt.*, I'm glad to get rid of them. Now you'll come back to the Baptist Church and stick by me your one and only lawful wife, *Engineering.* Be strictly mathematical and we shall live long and prosper.

(Loud knocking at door)

Newt. Mechs.: Oh . . . . (Shivers)

(Knocking repeated)

Engineering: Come in. (Knocking repeated. She goes to door and opens it.)

(Powerful stranger presents himself.)

Engineering: Who are you?

Geometry: I am called *Geometry,* I am the sheriff. See my badge. (He displays a badge marked with a large circle)

Newt. Mechs.: Oh . . . . (Covers his face with his hands) Oh . . . . .

Engineering: What do you want? A certain geometry, *Elementary* by name, used to serve us well.

Geometry: That was my infant self. I have come for *Mechanics.* (boastfully) See how big I am.

Newt. Mechs.: Woe! Woe! Woe!

Geometry: (to *Newtonian Mechanics*) I shall put you in my pocket. I shall consume you alive. (Declaims) Yes, Yes, *Geometry* will swallow up *Mechanics.* Live *Geometry,* down with *Mechanics.*

Newt. Mechs.: (Weeps out loud) Oh . . . . .

Geometry: Ha, you sniveling creature. Get in my pocket. In with you, old working man. You infinitesimal. In with you, I say.

Engineering: Now, *Master Geometry,* you are a big fine fellow there is no doubt about that. How you have grown! But away with you; put on your elementary aspects; leave *Mechanics* with me. The world cannot get on without me and I need this old classical fellow. (Aside) Cheer up

Newt., it's all right. (To Geometry again) I'll take him to *Physics* lots of times. I'll promise, We old folks need a girl like that. Maybe she'll come back week ends. These girls will run after handsome strangers. We'll behave ourselves. *Mathematics* will regulate our every move. Be a generous fellow, *Geometry*. Leave us alone now.

Newt. Mechs.: Oh leave us alone, leave us alone, Oh, Oh, Oh . . . . .

Geometry: Well, *Madam Engineering*, if you want the old fellow you may keep him. Possibly all of us can use him some. I'm quite sure that *Astronomy* and *Physics* can. As for me; I'll keep coming back and shall polish him up from time to time. He'll get on well enough but after a while you'll find that I have him completely rewritten in geometric language, and that (declaims) verily *Geometry* has swallowed up *Mechanics*. If you expect to use him then you, yourself, must know my higher dimensions. You too must come to *Geometry*, *Madam Engineering*, and I'll not wear my elementary aspects. No indeed there'll be no limit to the dimensions I'll wear. Adieu *Madam Engineering*.

Geometry: (To *Newtonian Mechanics*) Au revoir, old fellow. Good luck, but I'll be back. I'll have you in my pocket, do you understand, but for the present au revoir, auf wiedersehen. (Exit) (*Engineering* and *Newtonian Mechanics* fall into each others arms weeping.)

CLUB ACTIVITIES

Albion College Mathematics Club, Albion, Michigan.

Officers: President, Allene Day; Vice-President, Grace Ulbright; Secretary-Treasurer, Alberta Wocholz.

The programs for the year 1929–1930 were as follows:

October: Roll call—geometric figures in everyday life; A report of "The Cultural Value of Mathematics," by Aubrey J. Kempner.

November: Roll call—famous mathematicians and their work. Two reports," Mathematics in the junior high schools" and "The value of the history of mathematics in teaching."

December: A lecture by Dr. L. A. Hopkins, secretary of the College of Engineering and Architecture of the University of Michigan. Open to the student body.

January: Roll call—geometric theorems. A report, "Graded algebraic abilities in teaching"; and a contest conducted on an "Ancient duodecimal system."

February: A special report and discussion on "The plotting of the cubic."

March: Roll call—algebraic and geometric formulas. Two reports, "Mathematics in England and Germany" and "Mathematics in everyday life."

April: Roll call—geometric figures in nature. A report, "Watching the meteors."

May: Professor Evarts of Kalamazoo College gave a talk on "Uncomputative mathematics."

(Report by Alberta Wocholz)

Napierian Club of De Pauw University, Greencastle, Indiana.

Officers for the year 1929–1930 were: President, Charles Stunkel ('30); Vice-President, Josephine Newkirk ('30); Secretary, Margery Joslin ('30); Treasurer, Roy Holwager, ('30).

The following programs were given at the regular monthly meetings:

October 3, 1929. Election of new members.

October 10. "History of the Napierian Club and the life of John Napier" by Josephine Newkirk.

November 14. "Scales of notation" by Richard Jay; Book review of "Flatland" by Margery Joslin.

December 12. "Vector analysis" by Professor R. W. Babcock.

January 9, 1930. "Problems of a high-school teacher" by Christine Dietrick.

February 13. "History of algebra" by Mary McCord; "Probability in gambling" by Howard Fetters.

March 13. "The system of homogeneous coordinates" by Professor W. C. Arnold.
April 10. "Discussion of the new planet" by Roy Holwager; "Discussion of the coming eclipse of the moon and that of the sun" by Horace Barnett; Observations from McKim Observatory.
May 8. "Mathematics in music" by Professor H. E. H. Greenleaf; Election of new officers.

(Report by Margery Joslin)

Irrational Club of the University of Wyoming, Laramie, Wyo.

The officers for the year 1929–1930 were: President, Robert Hill; Vice-President, Imogene Montgomery; Secretary-Treasurer, Rowene Danielson; Faculty Advisor, Greta Neubauer.

The programs of the club for the year 1929–1930 were as follows:

October 9, 1929. Election of officers. Fall party.
November 7. "Development of the nine point circle" by Miss Montgomery.
November 21. "Mathematical cross-word puzzle" by club officers; "Graphical solutions for the complex roots of simultaneous quadratic equations" by Professor C. F. Barr.
December 5. "Life history of Réné Descartes" by Mr. Call; "Relation of mathematics and architecture" by Mr. Hitchcock.
January 16, 1930. "Logarithms" by Miss Neubauer; "Mechanical demonstrations of logs and cosines" by Robert Hill.
January 30. "Mathematics in music" by Mrs. Shaw; "The Irrational Club Song" by Mr. Barr.
February 21. "The Simpson line" by Miss Achenbach; "The problem of Apollonius" by Mrs. Shaw; "The locus of the point of similitude of two similar triangles, similarly placed, during rotation of equal amounts about dissimilar points" by Miss Montgomery and Miss Wortheim; A mechanical device showing the locus of points of similitude was demonstrated by Mr. Mize.
March 7. "History of the slide rule" by Miss Neubauer; "The slide rule" by Professor Goodrich.
March 21. "Determining pi by tossing matches" by Mr. Warner.
April 10. "Astronomy and mathematics" by the astronomy class.
May 8. "History of pi" by Rowene Danielson; "History of the equals sign" by Ada Burke; "History of e and of logarithms" by Grace Reid.
May 26. Annual beefsteak fry held in the mountains of the Laramie Range.

(Report by Rowene Danielson)

Mathematics Club of the College of the City of Detroit, Detroit, Michigan.

Any student of mathematics, or for that matter any student interested in mathematics at the College of the City of Detroit, may become a member of the club by simply coming to the meetings. The organization is rather loose, hence the president's job is a responsible one. Although no time is specified for meetings they have generally been held at least once a month, at the discretion of the president. The feature of each meeting is a paper, relating to some branch of mathematics, presented by a student. The widely varying nature of these papers is made apparent in the following paragraphs.

The first meeting of the year 1929–30 was held in October, and at that time Clarence Wylie was elected president. Immediately following the election, Mr. Wylie presented a paper on "Three dimensional plane geometry," in which he treated the imaginary element in plane analytics as a third dimension using i as a revolving operator.

Early in December, Gordon Wilcox presented a paper on "Conditions for exact division," in which he treated the relations between the digits of a number and the remainder or lack of remainder when divided. Here Mr. Wilcox introduced an interesting application of certain series.

At the April meeting the writer presented a paper on "Determinants," in which the algebra and elementary properties of this useful device were dealt with. Later in April Clarence Wylie discussed "Elementary analytic projective geometry"; he took up such topics as the principle of duality, homogeneous coordinates, and transformations involving the line at infinity.

Early in May, Miss Jean Persons, a graduate of the college teaching in the Detroit Public Schools, spoke on "Geometrical constructions by paper folding." Later in the same month Kenneth E. Stecker presented a paper on the "Approximate division of a circumference." This was the final meeting of the year.

(Report by Glen Brookes)

The Undergraduate Mathematics Club of the University of Iowa, Iowa City, Iowa.

Because of the untimely death of the president of the club, Mr. Wilbur B. Elliff, the first meeting of the club was not held until the second semester. At that time the following officers were elected for the remainder of 1929–30 and also for 1930–31: Professor L. E. Ward, Faculty Adviser; Mr. Carl H. Fischer, President; Miss Thelma Coate, Secretary-Treasurer.

The program for the semester was as follows:

February 20, 1930. "Methods of extracting square roots" by Professor L. E. Ward.
March 14. "A method of approximating to the *n*th root of a number" by Mr. Milton S. Weinberger.
March 27. ""Meteoric phenomena" by Mr. Lloyd O. Ritland.
April 11. "Confocal conics" by Miss Gerturde Rickey

(Report by Thelma Coate)

The Mathematics Club of the University of Georgia.

This informal club has no regularly appointed officers, no membership requirements, and no dues. By common consent it meets twice a month, and is attended by the faculties of the University of Georgia, State Teachers College, and Lucy Cobb Institute, together with the graduate students in mathematics. Those attending take turns as speakers. The programs for the year 1929–1930 were historical in character and D. E. Smith's *Source Book in Mathematics* served as inspiration and outline.

(Report by David F. Barrow)

The Mathematics Club of the Cooper Union Institute of Technology, New York, N.Y.

The officers for the year 1929–30 were: President, Leo Rubinowitz ('30); Vice-President, G. D. Champlin ('30); Secretary, David Rabinovich ('31); Treasurer, W. T. Eddy ('32); Assistant Secretary, J. J. Murphy ('31). The club has a paid membership of 170, the yearly dues being 25 cents.

The programs for the year were as follows:

Nov. 12, 1929. "Probability and its applications" by Mr. E. C. Molina, research engineer, American Telephone & Telegraph Co.
Dec. 3. "Application of the catenary to railway electrification" by G. D. Champlin ('30).
Dec. 17. "Elements of the theory of integers" by Fitzgerald Bramwell ('33.)
Jan. 14, 1930. "Bernoulli numbers" by Leo Rubinowitz ('30.)
Jan. 28. "The bearings of mathematics" by Professor C. J. Keyser, Columbia University.
Feb. 18. "Fundamental ideas of projective geometry" by Joseph Stitelman ('32).
Mar. 11. "The slide rule and its uses" by Professor W. E. Breckenridge, Stuyvesant High School.
Mar. 25. "Euler's problem" by Robert Wolins, '31; "The Russian peasant multiplication system" by L. A. Kenworthy ('31).
April 15. "Curious properties of numbers" by Professor H. W. Reddick. Election of officers.

(Report by C. H. Lehmann)

Bryn Mawr Mathematics Club of Bryn Mawr, Pennsylvania.

The program of the Bryn Mawr Mathematics Club for the year 1929–1930 was as follows:
October 16, 1929. "The definition of trigonometric functions by means other than geometric" by Dr. D. V. Widder.

November 5. "The motion of an electrically charged particle in a magnetic field, as developed by Dr. Richard Gans on the principles of vector analysis" by Ruth Unangst ('30), and "The Life of Evariste Galois" by Elizabeth Baker ('30).

Announcement was made at this meeting of a $10 prize to be awarded for the best undergraduate club report of the year.

November 26. "Regular polygons" by Dr. Anna Pell Wheeler, Bryn Mawr College.

December 18. "The history and computation of π" by Pauline Huger ('32), and "Homogeneous point coordinates" by Anne Lea Nicholson ('30).

January 7, 1930. "Groups—the theory being built up from facts observed of the group four" by Dr. Thomas, University of Pennsylvania.

February 11. "Isolation of the real roots of an equaton of degree n—Sturm's functions and Bowidan's" by Miss Lucile Anderson, Bryn Mawr College ('30).

March 18. "Inversions" by Constance Cole ('30), and "Fundamental propositions of algebra" by Agnes Hannay.

April 15. "Historical problems of the calculus of variations especially the brachistochrone of the Bernoulli brothers" by Gretchen Mueller ('32), and "Applications of the calculus of variations," by Mary Peters ('30).

May 13. "Postulates of a miniature geometry—indicating what a sizable geometry could be inferred without a continuity assumption" by Dr. Marguerite Lehr, Bryn Mawr College. Dr. Widder awarded the prize, referred to above (Nov. 5), to Miss Agnes Hannay for her report on "Fundamental propositions of algebra."

The officers for the year 1929–1930 were: Miss Agnes Hannay, President; Miss Janet Wise, Vice-President and Treasurer; Miss Ruth Unangst, Secretary.

The officers elected for 1930–1931 are: Miss Gretchen Mueller, President; Miss Ruth Unangst, Vice-President and Treasurer; Miss Pauline Huger, Secretary.

The club is composed of graduates, and undergraduates who have had or who are taking the second year mathematics course. Meetings were held from about 4:00 to 6:00 o'clock, the reports and discussions being followed by tea, food, and choice puzzles.

(Report by Ruth Unangst)

The Mathematics Club of the University of Alberta, Edmonton, Alberta, Canada.

The programs for the year 1929–1930 were as follows:

October 22, 1929. "Present day tendencies in secondary mathematics" by A. J. Cook and E. W. Sheldon.

November 6. "Units" by E. S. Keeping.

November 19. "Infinity" by Louise Miller.

December 3. "Congruences" by A. J. Cook.

January 14, 1930. "Superposition" by E. D. M. Williams.

January 28. "The vogue of statistics" by E. W. Sheldon.

Feburary 11. "Mechanisms" by J. S. Beggs.

Feburary 25. "Difficulties in secondary mathematics" by D. L. Shortliffe.

March 7. "Relativity and geometry" by I. F. Morrison.

(Report by E. W. Sheldon)

The Mathematics Club of Brown University, Providence, R.I.

The following programs were given in 1929–1930:

October 29, 1929. "The Martian counts on his fingers" by Lulu Amelia Vorleck (graduate); "Cocoanuts, sailors, and a monkey" by John Bernard Chaffee ('31.)

December 3. "Moving in four dimension" by John Otis Prouty ('31); "Nomograms" by David Moskovitz (graduate).

January 14, 1930. "Measuring the distances of the stars" by Clincton Harvey Currier, associate professor of mathematics, Brown University.

February 18. "Mathematics of the pyramid builders" by Mary Taylor ('30); "The golden section and Fibonacci series" by Harold Irving Brown ('30).

March 18. "Newton and the calculus" by Donald Leigh Fowler, Jr. ('31); "Leibniz and the calculus" by Enis Eva DeMagistris ('31).

April 22. "A mathematical cinderella" by Heinrich Wilhelm Brinkmann, assistant professor of mathematics, Harvard University.

May. Picnic.

Committee on Program: Professor Bennett; Mr. Thurston; Lulu Amelia Vorleck (graduate); Mary Taylor ('30); John Otis Prouty ('31); Donald Leigh Fowler, Jr. ('31).

Committee on Arrangements: Mr. Krall; Ruth Barden Eddy ('32); Mary Elizabeth Brooks ('31); James Benjamin Brown ('31); Delbert Swan Wicks, Jr. ('32).

PROBLEMS AND SOLUTIONS

EDITED *by* B. F. FINKEL, OTTO DUNKEL, *and* H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3469. *Proposed by V. M. Spunar, Chicago, Illinois.*

A constant length, PQ, is measured along the tangent at any point, P, on a curve; give a geometrical construction for the center of curvature of the locus of the point Q. Also if PQ' be measured equal to PQ in the opposite direction along the tangent, prove that the point P and the centers of curvature of the loci of Q and Q' lie in a straight line.

3470. *Proposed by F. L. Wren, George Peabody College for Teachers.*

If the hypotenuse of a right triangle be divided into n equal parts and the vertex of the right angle be joined to these points of equal division, then, if d_i be the length of the lines so drawn, we have

$$\sum_{i=1}^{n-1} d_i^2 = \frac{(n-1)(2n-1)}{6n} h^2,$$

where h is the length of the hypotenuse.

3471. *Proposed by W. R. Ransom, Tufts College.*

In assigning dormitory rooms, a college gives preference to pairs of students in this order:

$AA, AB, AC, BB, BC, AD, CC, BD, CD, DD,$

in which AA means two seniors, AB means senior and junior, etc. Determine numerical values to assign to A, B, C, D so that the set of numbers $A+A, A+B, A+C, B+B$, etc., corresponding to the order indicated above, will be in descending magnitude. Find the general solution, and also the solution in least integers.

3472. *Proposed by Morgan Ward, California Institute.*

Let S_n denote the sum of the nth powers of the roots of $F(x)=x^3-Px^2+Qx-1$, where P, Q are integers; p, a prime of the form $3n+2$ chosen so that $F(x)$ is irreducible modulo p, and μ, the least value of n such that $S_n \equiv S_0$, $S_{n+1} \equiv S_1$, $S_{n+2} \equiv S_2$, mod p. Prove that if n is not divisible by μ, $S_n \equiv 0 \bmod p$, when and only when $S_{3n} \equiv 3 \bmod p$.

3473. *Proposed by J. Rosenbaum, Milford, Connecticut.*

In the triangle ABC, the incircle is tangent to CA at D and to CB at E. Through a variable point P on DE, AP and BP are drawn meeting CB at X and CA at Y.

Find the envelope of the line XY.

3474. *Proposed by R. E. Gaines, University of Richmond, Va.*

From a point P on the ellipse $b^2x^2+a^2y^2=a^2b^2$ two chords are drawn, one through $(ae, 0)$ and the other through $(-a, 0)$; for what value of e and what position of P will these two chords trisect the area of the ellipse?

SOLUTIONS

3417. [1930, 157]. *Proposed by C. O. Williamson, The College of Wooster, Wooster, Ohio.*

Construct a square so that each side shall pass through a given point.

I. *Solution by E. M. Berry, Lynchburg College*

Let A, B, C, D, be the given points. Draw a line BL through B perpendicular to AC and so that BL is equal to AC. Let d be the line joining the points D and L. Draw b parallel to d through B, and draw a and c perpendicular to d through the points A and C respectively. The lines a, b, c, d, form the required square, since the distance between a and c is the same as the distance between b and d. This follows from the fact that the angle between a and AC is the same as the angle between b and BL and that the segments AC and BL are equal.

In general there are six solutions, for there are two solutions with A and C on opposite sides of the square and there are three ways of picking the pairs of points for opposite sides.

II. *Solution by R. Goormaghtigh, Bruges, Belgium*

The following is a solution of the generalized problem: *Construct a rectangle $ABCD$ similar to a given rectangle $MNPQ$, being given four circles tangent to the*

sides of $ABCD$ and placed, with respect to the corresponding sides, in given regions of the plane, compared with the center of the rectangle.

Let a, b, c, d be the centers and r_a, r_b, r_c, r_d the radii of the given circles, respectively tangent at $\alpha, \beta, \gamma, \delta$ to AB, BC, CD, DA; call m and n two numbers proportional to the sides MN and NP of $MNPQ$, MN and NP being homologous to AB and BC.

We first consider the case when the four circles are, with respect to the sides of $ABCD$, in the same regions of the plane as the center of the rectangle and when the rectangles $ABCD$ and $MNPQ$ are directly similar. If b', d' be the projections of b, d on AB and a', c' those of a, c on DA, the projections of $\delta db\beta$ and $\gamma ca\alpha$, respectively, on AB and DA are

$$AB = r_d + d'b' + r_b, \quad DA = r_c + c'a' + r_a.$$

But these projections are in the same ratio as m and n. Therefore, if, on the perpendicular from d on ac, we take dw equal to $m \cdot ca$, so that the right angle (ca, dw) has the same sign as $MNPQ$, and if, on db, we take dv equal to $n \cdot db$, the projection of wv on AB is

$$n \cdot d'b' - m \cdot c'a' = \rho = m(r_a + r_c) - n(r_b + r_d).$$

Hence the following construction:

If, on the perpendicular from d on ac, we take dw equal to $m \cdot ca$, so that the angle (ca, dw) has the same sign as $MNPQ$ and then on db a length dv equal to $n \cdot db$, the directions of DA corresponding to the problem are those of the tangents drawn from v to a circle having w as center and ρ as radius.

If dw' is taken equal to $m \cdot ca$ but so that the angle (ca, dw') has the same sign as $QPNM$, the construction gives rectangles $ABCD$ inversely similar to $MNPQ$.

If, further, some of the given circles are, with respect to the corresponding sides, in regions opposite to those containing the center of the rectangle, the signs of the corresponding radii will be changed in the foregoing construction.

In the special case of problem 3417, $r_a = r_b = r_c = r_d = 0$, $m = n = 1$, and the construction becomes the following:

On the perpendicular from d on ac take $w'd = dw = ac$; then the side BC of the required square passes through w or w' (two solutions for this selection of pairs of points).

Also solved by S. F. Bibb, J. D. Deutsch, P. S. Dwyer, V. F. Ivanoff, J. H. Neelley, William Orange, A. Pelletier, W. A. Rees, J. Rosenbaum, Hazel E. Schoonmaker, E. F. Sweet, and E. C. Westerfield.

3418. [1930, 157]. *Proposed by David H. Dodge, Los Angeles, California.* Prove that

$$\frac{7\sum n^6 + 5(p+1)\sum n^4 + p\sum n^2}{7\sum n^6 - 5(p-1)\sum n^4 - p\sum n^2} = \frac{n^2 + n + p}{n^2 + n - p}.$$

where

$$\sum n^6 = (6n^7 + 21n^6 + 21n^5 - 7n^3 + n)/42;$$

$$\sum n^4 = (6n^5 + 15n^4 + 10n^3 - n)/30;$$

and

$$\sum n^2 = (2n^3 + 3n^2 + n)/6.$$

Solution by Beatrice Aitchison, The Johns Hopkins University

Set

$$5\sum n^4 + \sum n^2 = f_1(n) \text{ and } 7\sum n^6 + 5\sum n^4 = f_2(n);$$

then the problem requires the proof of the identity

$$\frac{f_2(n) + pf_1(n)}{f_2(n) - pf_1(n)} = \frac{n^2 + n + p}{n^2 + n - p}.$$

It is easily found that $f_2(n) = (n^2+n)f_1(n)$, and this suffices to prove the equality above.

Also solved by S. A. Corey, Mabel S. Graham, A. Pelletier, Irwin Roman, and Margaret M. Young.

3419 [1930, 196]. *Proposed by the late F. P. Matz.*

Determine the law of force in order that the orbit be a Cassinian oval.

Solution by Norman Anning, University of Michigan

Assume that the center of attraction is at the center of figure. It is proved in treatises on dynamics that P, the force per unit mass, is given by the formula

$$P = h^2p^{-3}dp/dr, \tag{1}$$

where h is a constant and where p and r are connected by the pedal equation of the path. The Cassinian oval defined by the relation $PF_1 \cdot PF_2 = a^2$, where F_1 is the point $(-c, 0)$ and F_2 is $(+c, 0)$, has the equation

$$(x^2 + y^2)^2 - 2c^2(x^2 - y^2) + c^4 = a^4. \tag{2}$$

From this, by standard methods which need not be reproduced here, we find the "p-r" equation

$$2a^2pr = r^4 + a^4 - c^4. \tag{3}$$

Now, with equation (1) and with p expressed as a function of r, the rest is shop-work and leads to the result

$$P = 4a^4h^2r(3r^4 - a^4 + c^4)(r^4 + a^4 - c^4)^{-3}.$$

References: For (1), Norris & Legge, *Mechanics via the Calculus*, p. 274; for (2) and (3), Hilton, *Plane Algebraic Curves*, p. 324. For finding p-r equations, Edwards, *Differential Calculus*, p. 161.

Also solved by William Hoover.

3421 [1930, 196]. *Proposed by Otto Dunkel, Washington University.*

A convex polygon of n sides may be divided into triangles by its diagonals which intersect only at their extremities. Derive an expression for the number of ways in which this may be done.

I. *Solution by W. A. Bristol and W. R. Church, University of Pennsylvania*
Mr. Bristol's Contribution

Let K_n represent the number of ways a polygon of n sides can be divided in accordance with the conditions of the problem. Consider the vertices of any polygon as numbered consecutively from 1 to n. Draw diagonal $n2$ cutting off triangle $n12$, leaving a polygon of $n-1$ sides. This polygon of $n-1$ sides can be divided into triangles in accordance with the rules of the problem in K_{n-1} ways. Next, draw diagonal 13 and again the polygon of $n-1$ sides left can be divided into triangles in K_{n-1} ways, all different from the first set of K_{n-1} ways. Proceed to cut off successive vertices by diagonals joining each time the two vertices adjacent to the vertex to be cut off. The number of new ways of dividing the original polygon, for the ith diagonal, $i>2$, will be given by subtracting from K_{n-1} the sum of the first $i-2$ terms in the series (see Table 1 below) for the polygon of $n-1$ sides.

A triangle will be considered as being divided in one way. The above shows, as is also evident, that a quadrilateral can be divided in $1+1=2$ ways; a pentagon in $2+2+(2-1)=5$ ways. It is easy to see that the number of new ways added when $i=n-2$ will always be 1 and that there can be no new ways when $i=n-1$. Hence K_n is given by a series of $n-2$ terms which decreases after the second term according to the law stated in the above paragraph.

Thus we have

TABLE 1

n		K_n
3	1	1
4	1 + 1	2
5	2 + 2 + 1	5
6	5 + 5 + 3 + 1	14
7	14 + 14 + 9 + 4 + 1	42
8	42 + 42 + 28 + 14 + 5 + 1	132
...	.....	...

To illustrate further a point in the argument it will be noted that in obtaining the 5th term in the series for an octagon, the term is given by subtracting from K_7 twice K_6 and also K_6-K_5. But K_6-K_5 is the third term in the series for K_7.

Mr. Church's Contribution

Delete the first term in each series and renumber the rows according to the number of the row from top down. Next, reverse the order of terms in each deleted series, giving the following

TABLE 2

r \ c	1	2	3	4	5
1	1				
2	1	2			
3	1	3	5		
4	1	4	9	14	
5	1	5	14	28	42

Let ${}_cT_r$ represent the number in the cth column and rth row. The numbers ${}_cT_r$ are defined for all positive integral values of c and r when $c \leqq r$ by the following relations:

$$ {}_1T_r = 1, \qquad (r = 1, 2, 3, \cdots)$$

$${}_cT_r = \sum_{i=c-1}^{r} {}_{c-1}T_i.$$

By use of the following identity

$$\sum_{i=1}^{n} \prod_{j=i}^{i+x-1} j = \frac{1}{x+1} \prod_{k=n}^{n+x} k,$$

the following formula can be proved by induction:

$${}_cT_r = \frac{r-c+2}{r+1}\binom{r+c-1}{r}.$$

When $c=r$, we have

$${}_rT_r = \frac{2}{r+1}\binom{2r-1}{r}.$$

Now changing to the notation for polygons, where $r=n-2$, this becomes

$$K_n = {}_{n-2}T_{n-2} = \frac{2}{n-1}\binom{2n-5}{n-2}.$$

II. *Partial Solution by Vladimir F. Ivanoff, San Francisco, California*

Denote the vertices of the polygon by $P_1, P_2, \cdots, P_n$ and the number of combinations of diagonals that divide the polygon into triangles by A_n. Con-

sider the side P_1P_n. In every combination, P_1P_n belongs to a certain triangle $P_1P_iP_n$ $(1<i<n)$. Therefore, taking $i=2, 3, \cdots, (n-1)$, we obtain $(n-2)$ groups of combinations, which include all possible cases.

(1). If $i=2$, we have a group with the diagonal P_2P_n always present. This group includes as many combinations as there are in the $(n-1)$-gon $P_2P_3 \cdots P_n$, i.e., A_{n-1}.

(2). If $i=3$, we have the diagonals P_3P_1 and P_3P_n always present. This group therefore includes as many combinations as there are in the $(n-2)$-gon $P_3P_4 \cdots P_n$, i.e., A_{n-2}.

(3). If $i=4$, we have always the triangle $P_1P_4P_n$ present. The quadrangle $P_1P_2P_3P_4$ and $(n-3)$-gon $P_4P_5 \cdots P_n$ are adjacent to it. The former yields A_4 combinations, and the latter A_{n-3} combinations. Therefore, the number of all combinations in this group is $A_{n-3} \cdot A_4$. For the groups $(4, 5, \cdots)$. we have, respectively, $A_{n-4} \cdot A_5$, $A_{n-5} \cdot A_6$, etc., $\cdots$. For the group $(n-2)$, the number of combinations is the same as for the first one, i.e., A_{n-1}.

Thus

$$A_n = A_{n-1} + A_{n-2} + A_{n-3} \cdot A_4 + A_{n-4} \cdot A_5 + \cdots + A_5 \cdot A_{n-4} + A_4 \cdot A_{n-3} + A_{n-2} + A_{n-1}.$$

Since $A_3 = 1$ and $A_4 = 2$, we have

$$A_5 = A_4 + A_3 + A_4 = 5,$$
$$A_6 = A_5 + A_4 + A_4 + A_5 = 14,$$
$$A_7 = A_6 + A_5 + A_4 \cdot A_4 + A_5 + A_6 = 42,$$
$$A_8 = A_7 + A_6 + A_5 \cdot A_4 + A_4 \cdot A_5 + A_6 + A_7 = 132,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Also solved by A. G. Clark.

A Note by the Proposer. The partial solution above may be completed by the usual process of finding the generating function for the A_n's. For convenience set $A_2 = 1$; then we may write

$$A_n = \sum_{i=2}^{n-1} A_i A_{n-i+1}.$$

Now define in a purely formal manner the generating function $f(x)$ so that

$$f(x) = \sum_{n=2}^{\infty} A_n x^{n-2}.$$

Hence

$$[f(x)]^2 = \sum_{n=3}^{\infty} x^{n-3} \sum_{i=2}^{n-1} A_i A_{n-i+1} = \sum_{n=3}^{\infty} A_n x^{n-3}$$
$$= [f(x) - 1]x^{-1}.$$

Thus

$$f(x) = [1 - (1 - 4x)^{1/2}](2x)^{-1}$$

is the solution of the quadratic which gives $f(0) = 1$. If we regard x as restricted to the values $|x| \leqq a < 4^{-1}$, then we may develop $f(x)$, as defined above, in an absolutely convergent power series. With this restriction, all the formal work above is legitimate and $f(x)$ is our generating function for the A_n's. By expansion of $f(x)$ we have

$$f(x) = \sum_{n=2}^{\infty} \frac{2}{n-1}\binom{2n-5}{n-2}x^{n-2};$$

and hence

$$A_n = \frac{2}{n-1}\binom{2n-5}{n-2}.$$

The argument for the last part of the first solution may be put in the following form. As stated ${}_cT_r$ is completely defined by the two given requirements. The values of ${}_3T_r$ and ${}_4T_r$ are easily derived in turn from these two requirements. The form of these results suggest a function of the form

$$f(c, r) = (r - c + 2)\frac{(r + c - 1) \cdots (r + 3)(r + 2)}{(c - 1)!} = \frac{(r - c + 2)}{r + 1}\binom{r + c - 1}{c - 1}.$$

In order to prove the identity $f(c, r) = {}_cT_r$, we have merely to verify that $f(c, r)$ satisfies the two requirements. Now $f(1, r) = 1$ by the usual definition of the factorial symbols. Also we easily find that $\Delta f(c, r) = f(c-1, r+1)$, where Δ applies to r alone. Hence

$$\sum_{i=c-1}^{r} f(c - 1, i) = \sum_{i=c-1}^{r} \Delta f(c, i - 1) = f(c, r) - f(c, c - 2), \quad c \geqq 2,$$
$$= f(c, r).$$

This completes the proof.

3422 [1930, 197]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Prove that the polar reciprocal of a sphere with respect to a given sphere is a quadric of revolution. Discuss the nature of this quadric.

Solution by J. H. Neelley, Carnegie Institute of Technology

Let us find the polar reciprocal of the sphere S: $x^2+y^2+z^2=a^2$ with respect to the sphere S': $(x-b)^2+y^2+z^2=r^2$. The tangent plane of S at $P_1(x_1, y_1, z_1)$ and the polar plane of its pole $P'(x', y', z')$ as to S' being identical we have

$$x_1:(x' - b) = y_1:y' = z_1:z' = a^2:(r^2 - b^2 + bx').$$

Solving for x_1, y_1, and z_1 and substituting in the relation $x_1^2+y_1^2+z_1^2=a^2$ we have the required polar reciprocal with equation

$$(a^2-b^2)x^2-2b(a^2-b^2+r^2)x+a^2(y^2+z^2)=(r^2-b^2)^2-a^2b^2.$$

This is obviously a quadric of revolution.

The three parameters a, b, and r may be varied so that the quadric is either a sphere, a circular paraboloid, a hyperboloid of one sheet, a hyperboloid of two sheets, an ellipsoid, or a cone. Of course an axis of the quadric is the line of centers of S and S' in all cases.

3424 [1930, 197]. *Proposed by Morgan Ward, California Institute of Technology.*

Let A and B be two permutable elements of an abstract group, of orders a and b respectively, and let c be the order of $C=AB=BA$. Show that m' divides c, and c divides m, where m is the L.C.M. of a and b, and m' is the product of all those prime factors of m which appear in a and b raised to different powers, every such factor being raised to the power to which it appears in m.

Solution by Wendell Holmes Langdon, Yale University

Since A and B are permutable, $(AB)^m=A^mB^m=1$, whence c is obviously a divisor of m.

Now let

$$a=fp_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r},$$
$$b=fp_1^{\beta_1}p_2^{\beta_2}\cdots p_r^{\beta_r}$$

be the decomposition of a and b into primes, where f contains those prime factors which are raised to equal powers in a and b, and $\alpha_i\neq\beta_i$, $\alpha_i\geqq 0$, $\beta_i\geqq 0$. Let

$$\gamma_i=\max(\alpha_i,\beta_i);$$

then

$$m=fp_1^{\gamma_1}p_2^{\gamma_2}\cdots p_r^{\gamma_r}$$

and

$$m'=p_1^{\gamma_1}p_2^{\gamma_2}\cdots p_r^{\gamma_r}.$$

Since c is a divisor of m, we have

$$c=f'p_1^{\delta_1}p_2^{\delta_2}\cdots p_r^{\delta_r},$$

where f' is a divisor of f, and $\delta_i\leqq\gamma_i$. In order to prove the theorem we have only to show that $\delta_i=\gamma_i$. Let us suppose that $\delta_i<\gamma_i$ for some i, say $\delta_1<\gamma_1$, and let us assume $\alpha_1>\beta_1$.

Now

$$(AB)^c=A^\sigma B^\sigma=1, \text{ where } \sigma=f'p_1^{\delta_1}\cdots p_r^{\delta_r},$$

We raise this equality to the power k, where

$$k = fp_1^{\mu}p_2^{\beta_2}p_3^{\beta_3}\cdots p_r^{\beta_r}$$

and

$$\mu = 0 \text{ if } \delta_1 \geqq \beta_1,\ \mu = \beta_1 - \delta_1 \text{ if } \delta_1 < \beta_1.$$

In the resulting equality, the exponent of B is obviously divisible by b, and we obtain

$$A^{\tau} = 1, \text{ where } \tau = p_1^{\delta_1+\mu}p_2^{\delta_2+\beta_2}\cdots p_r^{\delta_r+\beta_r}.$$

whence, a being the order of A,

$$\delta_1 + \mu \geqq \alpha_1, \text{ or } \mu \geqq \alpha_1 - \delta_1 > 0.$$

For $\mu=0$, this relation is impossible; for $\mu=\beta_1-\delta_1$, we obtain $\beta_1 \geqq \alpha_1$, which is contrary to hypothesis. Thus $\delta_i=\gamma_i$ and c is divisible by m'.

Also solved by the Proposer.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Dr. H. D. Curtis, director of the Allegheny Observatory, has been appointed director of the new observatory and head of the department of astronomy at the University of Michigan.

Dr. Lena J. Hawks has been appointed head of the department of mathematics at the Georgia State Womans College.

Mr. George Karelitz, of the Westinghouse Electric and Manufacturing Company, has been appointed associate professor of mechanical engineering at Columbia University.

Assistant Professor E. G. Keller, of the University of Texas, has been appointed a mathematician of the General Electric Company.

Mr. J. H. Kusner has been appointed assistant professor of mathematics at the University of Florida.

Professor Alfred Landé, of the University of Tübingen, has been appointed professor of theoretical physics at Ohio State University.

Professor H. F. Lusk, of the College of the Pacific, has been appointed professor of engineering mathematics at Sacramento Junior College.

Edgar D. Meacham, professor of mathematics and Assistant Dean of the College of Arts and Sciences, University of Oklahoma, is spending his sabbatical year in study at the University of Chicago.

Dr. Dorothy McCoy has been appointed professor of mathematics at Belhaven College.

Mr. W. L. Porter, of Rice Institute, has been appointed professor of mathematics at the Texas Agricultural and Mechanical College.

Associate Professor J. C. Slater, of Harvard University, has been appointed head of the department of Physics at the Massachusetts Institute of Technology.

Miss Mildred E. Taylor has been appointed professor and head of the department of mathematics at Mary Baldwin College.

Dr. J. M. Thomas, assistant professor of mathematics at the University of Pennsylvania, has been appointed assistant professor of mathematics at Duke University.

Dr. L. H. Thomas, of Trinity College, has been appointed associate professor of theoretical physics at Ohio State University.

Dr. E. L. Thompson has been appointed assistant professor of mathematics at Fisk University, Nashville.

Professor R. A. Thornton, of Kittrell College, has been appointed head of the department of mathematical sciences at Talladega College.

Mr. C. E. Van Horn has been appointed head of the department of mathematics at Fisk University, Nashville.

Dr. H. S. Wall, of Northwestern University, has been promoted to an assistant professorship of mathematics.

Dr. Marie J. Weiss has been appointed assistant professor of mathematics at Sophie Newcomb College, Tulane University.

Dr. Charles Wexler has been appointed head of the department of mathematics at the Arizona State Teachers College.

Dr. G. T. Whyburn has been appointed associate in mathematics at Johns Hopkins University.

The following appointments to instructorships are announced:

University of Alabama: Mr. H. S. Thurston.

University of Arkansas: Mr. Paul Cramer.

Colorado School of Mines: Mr. Myron C. Pawley.

Long Island University: Mr. A. J. Smith.

Professor W. H. Bristol, inventor and formerly professor of mathematics at Stevens Institute of Technology, has died at the age of seventy.

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The prize will be awarded hereafter every three years. The last award was in December, 1929, to Professor T. H. Hildebrandt. The next award will be in December, 1932, for the period of 1929-1931.

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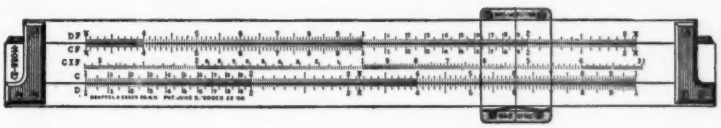

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

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BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fifteenth Summer Meeting of the Association, Minneapolis, Minnesota, Sept. 7-8, 1931.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1931.

ILLINOIS.
INDIANA.
IOWA.
KANSAS—Topeka, Kansas, Jan. 24.
KENTUCKY.
LOUISIANA-MISSISSIPPI.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA.
MICHIGAN.
MINNESOTA.
MISSOURI.
NEBRASKA.
OHIO—Columbus, Ohio, April 2.
PHILADELPHIA—Philadelphia, Pa., Nov. 28.
ROCKY MOUNTAIN.
SOUTHERN CALIFORNIA.
TEXAS—Fort Worth, Texas, Jan.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE FOURTEENTH ANNUAL MEETING OF THE MISSOURI SECTION

The fourteenth annual meeting of the Missouri Section of the Mathematical Association of America was held at the University of Missouri, Columbia, Mo. on Friday morning, November 28, 1930. The session was presided over by the chairman, Professor Louis Ingold, and by the vice-chairman, Professor Eugene Stephens, who took the chair while Professor Ingold presented a paper.

The meeting was followed by sessions of the American Mathematical Society on Friday afternoon and Saturday morning. The Missouri Alpha Chapter of Pi Mu Epsilon entertained with a tea on Friday afternoon, and on Friday evening a dinner was held at the Hotel Tiger.

There were present eighteen persons, including the following fifteen members of the Association: R. S. Christian, R. R. Fleet, Byron Ingold, Louis Ingold, G. H. Jamison, Paul Muehlman, W. O. Pennell, A. D. Pierson, P. R. Rider, W. H. Roever, Eugene Stephens, G. B. Sweazey, G. E. Wahlin, W. D. A. Westfall, E. Kathryn Wyant.

The following officers were chosen for the coming year: Chairman, W. H. Roever, Washington University; Vice-chairman, A. D. Pierson, Kansas City Junior College; Secretary-Treasurer, P. R. Rider, Washington University. It was decided to hold the 1931 meeting at Washington University, St. Louis, at the time of the meeting of the Missouri State Teachers Association in November.

The following program was presented:

I. *Papers*

1. "A visualization of homogeneous coordinates" by Professor W. H. Roever, Washington University.

2. "External Brocard points of a triangle" by Professor Louis Ingold, University of Missouri.

II. *A Symposium on Advanced College Courses in Algebra*

3. "Advanced algebra for the undergraduate" by Professor G. H. Jamison, Northeast Missouri State Teachers College.

4. "Algebra as an instrument of research" by Professor G. E. Wahlin, University of Missouri.

5. General discussion.

Professor Roever's paper was discussed by Professor Louis Ingold. Professor Jamison's paper led to an interesting discussion by Professors Stephens, Westfall, and Louis Ingold, and Mr. Pennell. Professor Wahlin's paper was commented on by Professor Rider. The general discussion of the symposium was participated in by Professors Fleet, Westfall, Stephens, Wahlin, and Wyant.

Abstracts of the papers follow:

1. In this paper Professor Roever showed (1) how a student may be introduced to the notion of homogeneous coordinates by citing the familiar example